ECHNOLOGY DEPT.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 61 NUMBER 3

PUBLIC LIBRARY
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MARCH 1954

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

Carl B. Allendoerfer, *Editor*

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NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

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Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

OF COURSE AND COURSES*

SAUNDERS MacLANE, The University of Chicago

Mr. President, members of the Assocation, and guests. I am here to retire, but I do not wish to be retiring. Both ends can be accomplished if I address myself to the principal objective of the Mathematical Association of America: that of the steady and imaginative improvement of collegiate education in Mathematics.

Collegiate education in Mathematics must not be construed too narrowly. On the one hand, we cannot ignore high school mathematics; this is the source of our entering students with their manifold difficulties in Mathematics. On the other hand, we cannot neglect graduate instruction in Mathematics; this is the source of our new teachers and the goal of some of our ablest students.

Collegiate education in Mathematics needs the most imaginative and vigorous reform, for it is now beset by numerous troubles and inadequacies.

These are internal troubles. Many of our courses cleave valiantly to a weak and obsolete tradition. Calculus has a perspicuous and beautiful intellectual structure, but its usual presentation is distorted by the unhappy fact that each new "standard" calculus text must copy the weakness of a long line of equally imitative predecessors. Trigonometry is in worse state; the publishers and authors of trigonometry texts conspire to demonstrate in exhaustive detail the combinatorial fact that an infinity of different texts is possible. The demonstration can be given more briefly: just combine all the alternative orderings of the following: define first the trignometric functions of an acute angle, or first those of a general angle; first triangles by logarithms or triangles by natural functions; identities first or equations first, *etc.*, *etc.*, *etc.* Among the standard courses, College Algebra is in perhaps the worst shape. Years ago, when it was first established†, it perhaps made sense, but under the pressure from masses of weaker and weaker entering students, the course has been analytically continued by continued dilutions. Today "College Algebra" stands for a subject which ought to be taught in the high school and which has nothing to do with algebra.

These are only some of the internal troubles of collegiate mathematics. There are also numerous external troubles. On the one hand, the social scientists have discovered that Mathematics can be of use. They properly complain that calculus and linear algebra are currently taught with a view only to the engineering and physical applications. They urge needed reform, and they are under a great temptation to urge too much in the way of special courses addressed primarily to the social scientists. They may fall thereby into the basic error of assuming that one can prepare students to make the necessary new applications

* Retiring Presidential Address before the Mathematical Association of America, December 31, 1953.

† The title "College Algebra" is an old one; there are American books under this title by Thomson and Quimby (1880), Bowser (1888) and J. M. Taylor (1889)—and perhaps others earlier.

of novel mathematical ideas by training them primarily in the old applications of older ideas. It cannot be too often reiterated that the aim of collegiate mathematics is the understanding of mathematical ideas *per se*. The applications support the understanding, and not vice versa.

Mathematics, like the rest of the academic community, is also bedevilled by the current fashion of general education. In its basic assumption this fashion was necessary. The emphasis on the great books was a needed corrective to our sorry adherence to text books that were far from great. The emphasis on an education that is broad and common to all students was a necessary answer to the American problem of giving a mass education to masses of students barely removed from illiteracy. However, the general educators cannot avoid one basic observation about the intellectual status of our age: there in fact is no common and accepted conceptual organization of present day knowledge, hence there is no possibility of constructing courses to convey this organization. For this reason and others, the fashion of general education has now been over-extended. As evidence, I cite you the fact that every University president treats general education as the Russians treat science: each president claims that his university discovered it first, or at any rate discovered its only true realization. These enthusiastic presidents miss the point that general education was probably started without benefit of presidents, as at Columbia, which in effect had no president, and as at Chicago, which had a Chancellor and not a president.

In the face of these fashions and troubles, we must keep our minds fixed on the real objectives of collegiate education in Mathematics. We must contrive ever anew to expose our students—be they general students or specialized students—to the beauty and excitement and relevance of mathematical ideas. We must set forth the extraordinary way in which mathematics, springing from the soil of basic human experience with numbers and data and space and motion, builds up a far-flung architectural structure composed of theorems which reveal insights into the reasons behind appearances and of concepts which relate totally disparate concrete ideas.

The program of our sessions today will amply cover the various problems which here arise for the more elementary collegiate courses. I therefore propose to turn my attention to the problems which arise in advanced undergraduate and beginning graduate courses. It is here that the sweeping reforms are necessary; the curriculum in advanced mathematics must be so overhauled that it can set forth the real structure of Mathematics as it is today. This structure can no longer be presented by piecemeal courses, for it is simply no longer true that advanced mathematics can be split neatly into compartments labelled "algebra," "analysis," "geometry," and "applied mathematics."

The fact that these subdivisions have ceased to be relevant may be seen most strikingly by observing how many of the most significant current discoveries in Mathematics refuse to be classified in the old compartments. Basic problems in functions of several complex variables have recently been solved by

using the notion of a *faisceau* or "sheaf"—a notion coming straight from algebraic topology. Hilbert's fifth problem has been solved—using techniques derived from topology, from Hilbert-space theory, and from Lie groups. In which compartment does this lie? In algebra, some of the most fruitful advances arise because of the demands for new algebraic tools for use in topology or in algebraic geometry. One is forced to the realization that algebra is not, and indeed never has been, an independent discipline. Modern analysis is replete with distributions, with rings of operators and with representations, so much so that it is difficult to say whether it is analysis, or a wholly new subject.

You may protest that I am talking about research and not about education, and that this should be the business of the Society, not the Association. I answer, first: The character and direction of current research is the best indication of the ideas which we ought to be teaching. I answer, second: One of our main responsibilities is that of training the research mathematicians of the future. American Mathematics has made tremendous strides forward in the last two decades; an essential ingredient in this advance has been the infusion of European mathematical talent. In the decades to come, we must produce a similar infusion on our own and from our own students.

In training research mathematicians the old ideals are not sufficient. When research mathematics was first developed in this country, it was necessary that the emphasis be just on the idea of research: something new, something unpublished, with no especial attention to the significance of the results or their place in Mathematics. Now it is necessary that the research be done with full appreciation of the significance of the various parts of our science, and with full availability of the techniques from the various disciplines which may become necessary even for the apparently most specialized problem.

My topic is "Of Course and Courses;" I mean "of the courses which may be constructed to produce an integrated course in mathematics." As an existence proof for such a course, I offer you the *Elements de Mathematique* by that dipsomaniac, Nicholas Bourbaki. He has achieved a Gallic version of a conceptual integration of mathematics; you may point out that he has paid a high price by way of abstraction and length, but I submit you that the goal is worth the price.

To be more specific, I would like to describe the mathematics curriculum which has been developed by the Department of Mathematics at the University of Chicago. This curriculum is intended to carry serious students of Mathematics from the beginning of the Junior year of college through a Master's degree in Mathematics. The objective is the ambitious one of providing a complete introduction to all those ideas which are basic to Mathematics in the sense that they occur in their several aspects in more than one part of Mathematics.

I must confess at once that our curriculum is still divided for administrative convenience into algebra, geometry, and analysis, but our emphasis is and should be on the use of each of these techniques in the other fields. Some subdivision is

necessary because our courses must appear in units of one quarter each, but we view these individual courses—and we hope that the students so view them—as part of a larger pattern.

In algebra we present a sequence of five quarter courses. The first of these is an introduction to abstract algebra by way of number theory, group theory, and vector spaces. That abstract notions require introduction by way of examples is clear; this is done. But abstract notions are not "hard." The young students like to think in these terms. It is a happy omen for the progress of civilization that here (and elsewhere in the curriculum) the beginning student takes more easily to abstractions and generalities than does his professor, who had to understand them the hard way.

The beginning algebra course also offers an ideal place for the introduction of the notion of a mathematical proof. This must be done, even for students who don't aim to be mathematicians. Algebra provides a better locus for this introduction than does the traditional plan of introducing the student to rigor and complex variables at the same time—for this type of introduction led to the assumption that rigor was the same thing as epsilontics. Bringing in rigor with algebra brings it in sooner and in simpler form.

The second and third algebra courses deal completely with vector spaces and linear transformations with especial attention to such topics as the invariant description of linear transformations (elementary divisors and the Jordan and rational canonical forms), the properties of quadratic forms (in particular, the principal axis theorem and its geometrical meaning), and finally the various relevant general notions, such as invariants, equivalence, and dual vector spaces. These topics belong early in the training of a mathematician. The proper treatment of calculus for functions of several variables requires vector ideas; the budding statistician and the coming physicist need them; modern analysis is unthinkable without the notion of linear dependence and all that flows from it. Throughout these courses the infusion of a geometrical point of view is of paramount importance. A vector is geometrical; it is an element of a vector space, defined by suitable axioms—whether the scalars be real numbers or elements of a general field. A vector is not an n-tuple of numbers until a coordinate system has been chosen. Any teacher and any text book which starts with the idea that vectors are n-tuples is committing a crime for which the proper punishment is ridicule. The n-tuple idea is not "easier," it is harder; it is not clearer, it is more misleading. By the same token, linear transformations are basic and matrices are their representations.

The fourth algebra course reverts to abstract algebra proper, with a treatment of rings, homomorphisms, ideals, groups, normal subgroups, the Jordan-Hölder theorem, and the Sylow theorems. The important notion of tensor product of groups and of spaces is often included and probably should always be there, for it is needed to understand modern geometry and modern algebraic topology. It is proved that every finitely generated abelian group is a direct sum of cyclic groups. Indeed, this theorem is an almost ideal example of an algebraic

"structure theorem." I wish that I could report that this theorem is always established as a special case of the corresponding theorem about modules over a principal ideal ring, but I must confess that the latter notion goes down hard—even though it is necessary as a connection between the groups treated in this course and the canonical forms for linear transformations from the previous course. In any event, abstract algebra is done with emphasis on the basic idea of homomorphism—the object of algebra is not just the study of a mathematical system *per se*, but of mappings of one such system into another. This idea is not restricted to algebra, and thus underlines again the conceptual unity of Mathematics.

The fifth course of the algebra sequence deals with the Galois theory—one of the most beautiful examples of a self-contained mathematical discipline, and one of the most convincing demonstrations of the power of the notions "homomorphism" and "automorphism." The course terminates with the basic structure theorems for linear algebra, which are the models for current developments in the structure of rings and in the study of rings of operators on a Hilbert space.

The geometry sequence consists of three courses. The first of these takes up analytic geometry, already treated to some extent in the calculus course, and carries it further. At the same time, analytic projective geometry appears, not in the flowery decadence which this subject reached in its American heyday, but as a necessary introduction of geometrical ideas of duality and of locus, and as a first demonstration that geometry starts with the space of ordinary experience but has the fertility to conceive new spaces representing and extending that experience.

The second geometry course deals with the foundations of geometry. The axiomatics of projective geometry, with the introduction of coordinates on the basis of these axioms, is one of the most beautiful instances of the power of axiomatic method, and at the same time emphasizes how geometry leads to algebra and how abstract notions like those of endomorphism have concrete meanings. The course continues with n-dimensional projective geometry—collineations, correlations, and the classification of hyperplanes. Finally, it turns to non-euclidean and inversive geometry, where these geometries are given in terms of subgroups of the projective group, thus illustrating again the relevance of group theory.

The third geometry course treats differential geometry. Here again we see the inadequacies of the "standard" course in this subject in comparison with the actual state of Mathematics. It is no longer sufficient to consider curves, surfaces, curvature, torsion, and first and second fundamental forms, all as an elegant application of the calculus. One must pay attention to differential geometry as it now is: with this in view, the course omits some of the more extended and uninteresting parts of the classical doctrine and instead provides an introduction to ideas of differential geometry in the large (the four vertex theorem, the theorem on turning tangents, *etc.*) and to the modern ideas of differentiable manifolds. The geometry on a surface is just not adequate if it is

done only "locally"—the consideration of changes of coordinate systems must build up to the notion of a differentiable manifold, and the study of the first and second differential form must lead to the notion of exterior differential forms on a manifold. The business of the young mathematicians is with ideas, and these are the ones he must meet in this field, the sooner, the better.

Finally I turn to analysis. The ideas of calculus are presented to freshmen and sophomores, as usual. We attempt to treat calculus with proper attention to rigor and more rapidly than is the custom, this by trimming some of the barnacles which have accumulated through the years. One just doesn't need an infinity of different applications of the definite integral! The calculus for several variables is a hard nut; for example, the proper treatment of Stokes' and Green's theorem really requires the notion of exterior forms; I must confess that we have not yet found a good way to introduce these ideas where they belong in calculus.

The final course in our calculus sequence covers various topics in advanced calculus. It starts with the idea of uniform convergence for series; this is then applied to establish the standard results for Fourier series. The idea of successive approximation is then introduced and used to provide existence and continuity proofs for differential and integral equations. The course terminates with a survey of the methods of complex variable theory up through contour integration, leaving the more sophisticated treatment of these topics for a later course on the subject.

The further reaches of analysis can no longer be treated in isolation from other topics in Mathematics; properly construed, they rest essentially upon algebra and topology and in turn fructify these subjects. Hence the student next takes a sequence of two courses in topology. The first of these, on point sets and metric spaces, starts with the basic algebra of sets, including cartesian products, and develops the cardinal and ordinal numbers and the technique of using Zorn's lemma in its various forms. Then comes a study of metric spaces, including completeness and compactness; the power of the notion of a metric is illustrated by showing how uniform convergence can be realized as convergence in the metric of suitable function space. This treatment of metric spaces presents the ϵ-δ technique of analysis in its proper setting and motivates the more general notions of topology.

The latter ideas are covered in the second course, starting with the definition of a topological space and continuing through the study of the separation axioms, connectivity, and compact spaces. The connection with metric spaces is re-established by means of metrization theorems. The possibility of various topologies on function spaces is used to illustrate the breadth and power of the general notion of a topological space, while the compactification theorems illuminate the processes of constructing new spaces from old ones. The course ends with the Weierstrass-Stone approximation theorem and, when possible, with a brief treatment of fundamental groups and of covering spaces. The last topic in particular has great merit, with application on the one hand to topo-

logical groups, and on the other to algebraic topology, where it serves as an introduction to the more general notions of fibre bundles.

On this basic array of topological instruments the student is then well prepared to take up the further essential topics in analysis; a systematic study of complex variable theory (one quarter) and an examination of measure theory and Lebesgue and Stieltjes integrals (one quarter). The traditional material of a course in real variables is thus subdivided and dispersed—as it should be. It appears in part in the more general notions of metric spaces, and in part in more advanced specialized courses.

This then is the curriculum which my colleagues at the University of Chicago have laid out to cover mathematical ideas on a broad front. For the student who aims to teach, it provides a sound knowledge of what mathematics is about. For the student going further in Mathematics, it provides a broad base and technical equipment for the attack on further ideas—functional analysis, topological groups, algebraic topology, differentiable manifolds, algebraic geometry, the structure of rings, Lie groups, and the rest. The basic requirement on a sound curriculum is precisely that it give the necessary background for these and other studies, and thereby exhibit the unity of Mathematics.

In so outlining the curriculum which has been set up at Chicago, I do not wish to claim that it is perfect or unique. It has some gaps (for example, an introduction to partial differential equations). Other quite different organizations of material could be made; indeed some of these organizations could better stress the relation between various ideas. I claim then no perfection, I submit rather that this is but one first approximation to the pressing need for basic mathematical training. One must design modern and coherent curricula, cleared of traditional impedimenta and providing rapid access to new and general ideas, so that they can then be applied in special domains. I submit that this objective is a vital one and urge that you go and do better.

MOTIVES AND TRENDS IN MATHEMATICS*

H. K. FULMER, Georgia Institute of Technology

1. The main idea of this paper. The purpose of this paper is to set forth the thesis that in basic discoveries in science generally, and in mathematics particularly, the ordinary motives of men do not seem to operate. For example, if we list as ordinary motives (1) the desire for personal prestige, (2) the wish to create something useful—that is, the utility motive—and (3) (strongest of all)

* Condensed from a paper presented at the annual meeting of the Southeastern Section of the Mathematical Association of America at Alabama Polytechnic Institute, March 13–14, 1953.

the desire for personal profit, we hope to show that these have not usually been associated with basic discoveries. Indeed, and this is really the main idea, we hope to show that there seems to exist an opposition, a sort of settled aversion, a repulsion between these ordinary motives and the will to recognize and create new ideas. The notion is rather closely associated with the unwritten law that no really great teacher is in the profession purely for the money. And in similar vein, others have speculated on how Newton or Euler would regard the forty-hour week with ideas given a project number and reduced to small sub-sections each in charge of a director.

The study is pertinent at the present time since most of our current research (aside from the defense effort) is sponsored with the sole motive: Will it pay? If in the past this motive has not produced, there is reason to doubt that it will produce in the future and that our elaborate set-ups for study will yield a return.

2. An example from history. To cite in the history of mathematics a simple illustration of the fact that utility and the creative urge have not as a rule been found together, consider the history of plane geometry in Egypt and Greece. Legend claims that the early development of geometry occurred in Egypt, the reason being that the priestly class of that country had the leisure to study it. In this environment it might have taken root and developed into a permanent science. But in response to the needs of government the study of geometry was diverted toward practical ends. Granted that our historical accounts of how land must be measured to apportion taxes and how washed-out boundaries must be replaced may be in considerable part fiction, nevertheless geometry came to have the utilitarian motive. It became, under the Egyptians, a geometry of rules without reasons. It was applied geometry—no theorems, no coherence, no plan. What was the result? We are familiar with it: the Egyptians, always with the motive of utility, produced no permanent geometry. They failed utterly to produce any semblance of a logical system in this field.

Now consider geometry several hundred years later in Greece. The Greek mind, particularly Euclid's, entertained no notion of profit or application. The Greeks were not concerned with usefulness. But they did possess an instinctive urge toward learning for its own sake, a craving to discover the reasons for things. And so, with no thought of utility, but with every thought of reason and logical structure, the Greeks produced a geometry that still endures, a work of surpassing strength and beauty.

3. Another historical example. Consider the mathematics of the Roman Empire. The record here is one of peculiar sterility, a desert, a stretch of hundreds of years without a single noteworthy advance in our field. The Romans added nothing to their scientific inheritance. No Roman mathematician has a place in history. Even the few historical attempts at kindly references are open to question, since the historians of that period were impressed with the necessity of pleasing their rulers. Over this long period of hibernation of mathe-

matical reasoning, what characteristic was dominant in Roman thought? The answer, as we all know, is that the Romans were a practical people, that their primary thought was utility. Thus they sought a practical mathematics, a geometry of immediate usefulness, an arithmetic that would solve the problems of the day. The Roman mind had no time for definitions, for logical structure, for theorems and proofs. Though these people had inherited riches in abundance, they had not inherited the inquiring spirit, the native curiosity that had distinguished the Greeks. They had inherited rather the practical viewpoint, the notion that there was no place except for the useful. And so they produced nothing, added nothing, and the Roman Empire affords another historical example of the barren results that seem inevitably to attend the utility motive. Here again we could point to an apparent inborn opposition between the spirit to create and the desire to use.

4. Motives as expressed by great mathematicians. It is useful, in our study, to examine the motives of history's great mathematicians as expressed by themselves. What were their motives? Did they consider whether their studies would be useful or profitable? We quote the well-known remark of Archimedes,

> "Every kind of art which is connected with daily needs is ignoble and vulgar."

Galileo said:

> "I am filled with infinite astonishment and also infinite gratitude to God that it has pleased Him to make me alone the first observer of such wonderful things, which have been hidden in all past centuries."

And the remark of Newton is well known:

> "I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."

And still considering motives, recall that Descartes, while yet a soldier, occupied all his leisure time examining new ideas, testing out his first faint notions of analytic geometry. Newton spent his declining years editing and improving his notes even though he had no intention of publishing them. These glimpses into the lives and thoughts of Archimedes, Galileo, Descartes, and Newton illustrate again the main thesis of this paper—the idea of conflict between profit and utility as against the patient speculative curiosity that marks the true pioneer of new ideas.

5. Mathematicians in government and industry. We conclude this paper with a forward look. A trend has set in that is difficult to assess, a trend that is new and growing. We refer to the increasing drift of men highly trained in mathematics into the realm of government and of business. Thirty years ago a man or woman specializing in mathematics had only one field open—the teach-

ing profession. With the exception of a few men employed in statistics by insurance companies practically no mathematicians found a place in industry. This meant that the colleges were able year after year to recruit the most talented and accomplished mathematicians from the graduate schools. How rapidly this is changing we all know full well. No longer do the colleges have a monopoly on the best brains; the colleges are fighting a losing battle against better paying government and still better paying industry.

A recent study of the membership of this Association shows that at least 10% are engaged in work other than teaching. In round numbers this group of non-teaching members totals 450 and roughly equals in number the Association's entire membership in our Southeastern States. Many of these mathematicians work in various divisions of the Federal government. Many others are in research bureaus of vast industrial organization.

Are their surroundings and general motivation of the sort which in the past have stimulated learning? Consider that in the main their studies are directed. Few are free to follow their own inclinations. The mathematics is definitely a means toward a technical end. Groups are highly organized with chiefs, assistant chiefs, directors and project managers. Perhaps in industry there is less organization than in the Federal government, but in industry the profit motive is frank and open. The job in industry, with rare exceptions, is not the advancement of natural science at all. The job is to develop, produce, and sell goods at a profit. Stated simply, the job in industry is to outwit one's competitors, to produce a better article at a lower price.

6. Contrast of motives and conclusion. Contrast the motives here with the motives of those who in history advanced the frontiers of knowledge. How remote from organizational matters, from chiefs, directors, and advisory councils, from projects and reports is the patient, inquiring, doubting, questioning attitude of a Galileo or a Newton. Could it be that there exists, as history seems to show, an implacable conflict between the selfish motive and the will to create? Could it be that in the long years of the future only meager returns will attend our vast and expensive research organizations, that man's progress still must wait on those who hear and heed the ancient call

> "Something lost beyond the ranges,
> Something hidden, go and find it."

A CHAIN OF CIRCLES ASSOCIATED WITH THE 5-LINE

J. W. CLAWSON, Ursinus College

1. Introduction. In the second half of the nineteenth century there was considerable activity in the field of elementary geometry. Many new points, straight lines and circles associated with the triangle were discovered. Some of the most interesting are found in current textbooks on Modern Geometry. In the same period investigations into the geometry of the complete quadrilateral or 4-line uncovered many interesting relations. The writer collected many of these in an article published in 1919. [1]. The complete pentagon or 5-line has not been so extensively explored. The important theorem that the mid-diagonal lines (Newtonians) of the five 4-lines obtained by omitting in turn each side of the 5-line concur at the center of the conic inscribed in the 5-line is implicit in a lemma of Newton [2]. The point of concurrency might well be termed the Newtonian point of the 5-line. Apart from this, few theorems are well-known except those connected with the Clifford and Morley chains. These are the basis for this article and are stated in section 2. In sections 3–5 we apply these theorems to the 5-line and add some connective material. In sections 6–8 we use the method of inversion to add some new points and circles to the geometry of the pentagon. The methods of pure Euclidean geometry are used throughout.

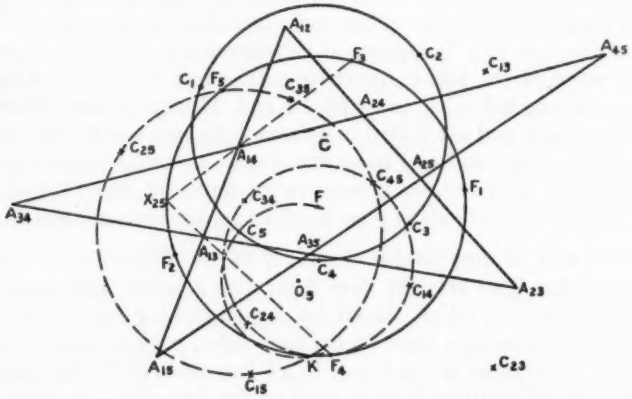
2. Clifford and Morley chains. In 1871, W. K. Clifford announced [3] a series of theorems: Two straight lines determine a point. Three straight lines determine a circle. In a 4-line, the circles circumscribing the four triangles obtained by omitting in turn each of the four sides, pass through a point—the Wallace [4] or focal point of the 4-line. In a 5-line, the five Wallace points of the 4-lines obtained by omitting in turn each of the five sides, lie on a circle, [5] which we shall call the Clifford circle of the 5-line. In a 6-line, the six Clifford circles of the 5-lines obtained by omitting in turn each of the six sides, pass through a point, which we shall call the Clifford point of the 6-line. And so on, so that in a $2n$-line, the $2n$ Clifford circles of the $2n-1$-lines obtained by omitting in turn each of the $2n$ sides, pass through the Clifford point of the $2n$-line; while in a $2n+1$-line, the $2n+1$ Clifford points of the $2n$-lines obtained by omitting in turn each of the $2n+1$ sides, lie on the Clifford circle of the $2n+1$-line.

In 1877, G. de Longchamps [6] stated the following series: In a 4-line, the centers of the circles circumscribing the four triangles, obtained by omitting in turn each of the four sides, lie on a circle which we shall call the Morley (circumcentric) circle of the 4-line; this circle also passes through the focal (Wallace) point. In a 5-line, the five centers of the Morley circles of the 4-lines, obtained by omitting in turn each of the five sides, lie on a circle which we shall call the Morley circle of the 5-line; while these five circles themselves pass through a point which we shall call the de Longchamps point of the 5-line. And so on, so

that in an n-line, the n centers of the Morley circles of the $n-1$-lines, obtained by omitting in turn each of the n sides, lie on a circle which we shall call the Morley circle of the n-line; while these n-circles themselves pass through a point which we shall call the de Longchamps point of the n-line.

Proofs of these theorems may be found in the original papers, more systematically in such books as Coolidge, "Treatise on the Circle and the Sphere" [7] and, by less elementary methods, in Morley and Morley, "Inversive Geometry" [8].

3. Clifford circle, Morley circle, de Longchamps point of the 5-line. We now re-state these theorems insofar as they apply to the 5-line, introducing the notation that will be used in this paper, and adding a theorem, (f) which was announced by F. Bath in 1940 [9].



Let l_1, l_2, l_3, l_4, l_5 be five straight lines in a plane, no two parallel and no three concurrent. Let A_{12} be the intersection of l_1, l_2; and so on. Let the circumcircle of $A_{12}A_{23}A_{13}$ be $\mathfrak{C}_{45}$, its center C_{45}; and so on. Then

(a) $\mathfrak{C}_{12}$, $\mathfrak{C}_{13}$, $\mathfrak{C}_{14}$, $\mathfrak{C}_{15}$ *have a common point, F_1.*

(b) C_{12}, C_{13}, C_{14}, C_{15}, F_1 *lie on a circle, $\mathfrak{C}_1$, its center C_1. There are five points like F_1 and five circles like $\mathfrak{C}_1$. F_p and F_q both lie on $\mathfrak{C}_{pq}$.*

(c) F_1, F_2, F_3, F_4, F_5 *lie on a circle, $\mathcal{F}$, its center F. This is the Clifford circle of the 5-line.*

(d) $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$, $\mathfrak{C}_4$, $\mathfrak{C}_5$ *have a common point, K. This is the de Longchamps point of the 5-line.*

(e) C_1, C_2, C_3, C_4, C_5 *lie on a circle, $\mathfrak{C}$, its center C. This is the Morley circle of the 5-line.*

(f) $\mathcal{F}$ *also passes through K.*

Since Bath's proof of (f) is difficult, it may be well to give a simple proof here. We use R. A. Johnson's notation [10]; PQR is the angle through which the

line PQ taken as a whole must be rotated counterclockwise in order for it to coincide with QR taken as a whole.

Taking K to be with C_{45} an intersection of $\mathfrak{C}_4$ and $\mathfrak{C}_5$, $F_4KF_5 = F_4KC_{45} + C_{45}KF_5 = F_4C_{34}C_{45} + C_{45}C_{35}F_5$.

But, since $\mathfrak{C}_{34}$ and $\mathfrak{C}_{45}$ intersect in A_{12} and F_4, $F_4C_{34}C_{45}$ is half the central angle $F_4C_{34}A_{12}$ of $\mathfrak{C}_{34}$ and hence equal to $F_4F_3A_{12}$. Similarly $C_{45}C_{35}F_5 = A_{12}F_3F_5$. So, adding, $F_4KF_5 = F_4F_3F_5$. Hence circle $\mathcal{J}$ passes through K.

4. Ten points on the Clifford circle. We add the theorem: *F_pA_{qr}, F_qA_{pr}, F_rA_{pq} concur at a point on $\mathcal{J}$. Here p may be any of the digits* 1, 2, 3, 4, 5; *q any different digit; r any third digit. There are thus ten possible combinations.*

To prove this theorem we shall need to use Clifford's proof of theorem (c) above.

Let F_3A_{14} and F_4A_{13} intersect at X_{25}. Consider the triangle $A_{13}A_{14}X_{25}$ with the points F_3 on $X_{25}A_{14}$, F_4 on $X_{25}A_{13}$ and first the point A_{13}, second the point A_{15} on $A_{13}A_{14}$. Then since, in the first case, circles $A_{13}A_{13}F_4$ or $\mathfrak{C}_{45}$ and $A_{13}A_{14}F_3$ or $\mathfrak{C}_{35}$ intersect in F_5, in the second case, $A_{13}A_{15}F_4$ or $\mathfrak{C}_{24}$ and $A_{14}A_{15}F_3$ or $\mathfrak{C}_{23}$ intersect in F_2, it follows from a converse of Miquel's theorem that the circle $F_3X_{25}F_4$ passes through both F_5 and F_2. This proves that F_2, F_3, F_4 and F_5 are concyclic.

Since the intersection of F_1A_{34} and F_2A_{14} may be proved in the same way to lie on circle $\mathcal{J}$, it follows that F_1A_{34}, F_2A_{14} and F_4A_{13} concur at a point on $\mathcal{J}$. This does not seem to have been noted before.

5. A connective relation. We add another theorem which does not seem to have been stated before:

(g) *The de Longchamps point and the center of the Clifford circle are inverse points with respect to the Morley circle.*

We shall need the following lemma: $F_pF_qF_r - C_pC_qC_r = F_pA_{qr}F_r$.

Proof. $F_4F_5F_3 = F_4X_{25}F_3 = A_{34}A_{14}X_{25} + X_{25}A_{13}A_{34} + A_{13}A_{34}A_{14}$. Now $A_{13}A_{34}A_{14} = A_{13}F_5A_{14} = C_{45}C_{25}C_{35}$, (since $\mathfrak{C}_{25}$, $\mathfrak{C}_{45}$ intersect in A_{13}, F_5 and $\mathfrak{C}_{25}$, $\mathfrak{C}_{35}$ in A_{14}, F_5) $= C_{45}KC_{35} = C_4C_5C_3$, (since $\mathfrak{C}_4$, $\mathfrak{C}_5$ intersect in C_{45}, K and $\mathfrak{C}_3$, $\mathfrak{C}_5$ in C_{35}, K).

Hence $F_4F_5F_3 = A_{24}A_{14}F_3 + F_4A_{13}A_{23} + C_4C_5C_3$.

Then $F_4F_5F_3 - C_4C_5C_3 = A_{24}A_{12}F_3 + F_4A_{12}A_{23} = F_4A_{12}F_3$. This is generalized above.

We shall now prove that the circle FC_5K is orthogonal to $\mathfrak{C}$. Let O_5 be the center of this circle.

Now $O_5C_5C = O_5C_5F + FC_5C$. But $O_5C_5F = \frac{1}{2}\pi - FKC_5 = \frac{1}{2}\pi - (F_5C_{45}K - F_5F_3K)$ since C_5, F are centers of circles passing through F_5, K. But $F_5F_3K = F_5F_3F_4 - KF_3F_4 = F_5F_3F_4 - KF_5F_4$. Hence $O_5C_5F = \frac{1}{2}\pi - F_5C_{45}K + F_5F_3F_4 - KF_5F_4$.

Also $FC_5C = C_4C_5C - C_4C_5F = \frac{1}{2}\pi - C_5C_3C_4 - C_{45}KF_5$, since C_4C_5 is perpendicular to KC_{45} and FC_5 to KF_5.

Thus $O_5C_5C=\pi+F_5F_3F_4-C_5C_3C_4-F_5C_{45}K-C_{45}KF_5-KF_5F_4$. However, $\pi-F_5C_{45}K-C_{45}KF_5=KF_5C_{45}$ and $KF_5C_{45}-KF_5F_4=F_4F_5C_{45}$. Also, by the lemma, $F_5F_3F_4-C_5C_3C_4=F_5A_{12}F_4$. Hence, $O_5C_5C=F_5A_{12}F_4+F_4F_5C_{45}$.

But $F_4F_5C_{45}=\frac{1}{2}\pi-F_5A_{12}F_4$, considering $\mathfrak{C}_{45}$. Thus $O_5C_5C=\frac{1}{2}\pi$. Since FC_4K may similarly be proved orthogonal to $\mathfrak{C}$, theorem (g) follows.

6. New systems of circles and points. Now let us use a circle of inversion $\mathfrak{K}$ with K as center and with any definite radius. The five circles $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$, $\mathfrak{C}_4$, $\mathfrak{C}_5$, will invert into a 5-line c_1, c_2, c_3, c_4, c_5 with c_1, c_2 intersecting in c_{12}, and so on. We then make the same constructions as in the original 5-line. The circumcircle of $c_{12}c_{23}c_{13}$ shall be $\mathfrak{d}_{45}$, its center d_{45}. Then

(a′) $\mathfrak{d}_{12}$, $\mathfrak{d}_{13}$, $\mathfrak{d}_{14}$, $\mathfrak{d}_{15}$ have a common point, g_1.

(b′) d_{12}, d_{13}, d_{14}, d_{15}, g_1 lie on a circle, $\mathfrak{d}_1$, its center d_1. There are five points like g_1 and five circles like $\mathfrak{d}_1$.

(c′) g_1, g_2, g_3, g_4, g_5 lie on a circle $\mathfrak{g}$, its center g.

(d′) $\mathfrak{d}_1$, $\mathfrak{d}_2$, $\mathfrak{d}_3$, $\mathfrak{d}_4$, $\mathfrak{d}_5$ have a common point, l.

(e′) d_1, d_2, d_3, d_4, d_5 lie on a circle $\mathfrak{d}$, its center d.

(f′) g also passes through l.

(g′) g and l are inverse points with respect to $\mathfrak{d}$.

We now invert this system of circles with respect to the same circle $\mathfrak{K}$, so that c_{12} goes back into C_{12}, and so on. We should recall that of a pair of inverse circles, the inverse of the center of one circle is the inverse of the center of inversion in the other circle [11]. We obtain at once the following theorems which apply to the original 5-line and its circles.

(A) *Circle $C_{34}C_{35}C_{45}$, which we shall call $\mathfrak{D}_{12}$, and three other such circles, $\mathfrak{D}_{13}$, $\mathfrak{D}_{14}$, $\mathfrak{D}_{15}$ have a common point, G_1.*

(B) *If the inverse of K with respect to $\mathfrak{D}_{12}$ is D_{12}, then D_{12}, D_{13}, D_{14}, D_{15} and also G_1 lie on a circle, $\mathfrak{D}_1$.*

(C) *G_1, G_2, G_3, G_4, G_5 lie on a circle, $\mathfrak{G}$.*

(D) *$\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, $\mathfrak{D}_4$, $\mathfrak{D}_5$ have a common point, L.*

(E) *If the inverse of K with respect to $\mathfrak{D}_1$ is D_1, then D_1, D_2, D_3, D_4, D_5 lie on a circle, $\mathfrak{D}$.*

(F) *$\mathfrak{G}$ also passes through L.*

(G) *If the inverse of K with respect to $\mathfrak{G}$ is G, then G and L are inverse points with respect to $\mathfrak{D}$.*

But the circle $\mathfrak{F}$ also passes through K, by (f). If the inverse of $\mathfrak{F}$ in $\mathfrak{K}$ is f, then for example the five circles $\mathfrak{C}_2$, $\mathfrak{C}_3$, $\mathfrak{C}_4$, $\mathfrak{C}_5$, $\mathfrak{F}$ invert into a 5-line c_2, c_3, c_4, c_5, f. Let the intersection of c_2 and f be f_2, and so on. Repeating the same constructions as above with this 5-line and denoting the circle circumscribing $f_3f_4c_{34}$ by f_{34}, its center f_{34}, and so on, we have

(a_1) $\mathfrak{f}_{34}$, $\mathfrak{f}_{35}$, $\mathfrak{f}_{45}$, $\mathfrak{d}_{12}$ have a common point, g_{12}.

(b_1) f_{34}, f_{35}, f_{45}, d_{12}, g_{12} lie on a circle $\mathfrak{d}'_{12}$, its center d'_{12}.

(c_1) g_{12}, g_{13}, g_{14}, g_{15}, g_1 lie on a circle $\mathfrak{g}_1'$, its center g_1'.
(d_1) $\mathfrak{d}_{12}'$, $\mathfrak{d}_{13}'$, $\mathfrak{d}_{14}'$, $\mathfrak{d}_{15}'$, $\mathfrak{d}_1$ have a common point, l_1.
(e_1) d_{12}', d_{13}', d_{14}', d_{15}', d_1 lie on a circle $\mathfrak{d}_1'$, its center d_1'.
(f_1) $\mathfrak{g}_1'$ also passes through l_1.
(g_1) g_1' and l_1 are inverse points with respect to $\mathfrak{d}_1'$.

If we now invert this system with respect to the same circle $\mathfrak{K}$, so that f_2 goes back into F_2 and so on, we obtain at once the following theorems which apply to the original 5-line and its circles.

(A_1) *Circles $F_3F_4C_{34}$, which we shall call $\mathfrak{F}_{34}$ and two other such circles $\mathfrak{F}_{35}$, $\mathfrak{F}_{45}$, together with $\mathfrak{D}_{12}$ have a common point, G_{12}.*
(B_1) *If the inverse of K with respect to $\mathfrak{F}_{34}$ is F_{34}, then F_{34}, F_{35}, F_{45}, D_{12}, G_{12} lie on a circle $\mathfrak{D}_{12}'$.*
(C_1) *G_{12}, G_{13}, G_{14}, G_{15}, G_1 lie on a circle, $\mathfrak{G}_1'$.*
(D_1) *$\mathfrak{D}_{12}'$, $\mathfrak{D}_{13}'$, $\mathfrak{D}_{14}'$, $\mathfrak{D}_{15}'$, $\mathfrak{D}_1$ have a common point, L_1.*
(E_1) *If the inverse of K with respect to $\mathfrak{D}_{12}'$ is D_{12}', then D_{12}', D_{13}', D_{14}', D_{15}', D_1 lie on a circle, $\mathfrak{D}_1'$.*
(F_1) *$\mathfrak{G}_1'$ also passes through L_1.*
(G_1) *If the inverse of K with respect to $\mathfrak{G}_1'$ is G_1', then G_1' and L_1 are inverse points with respect to $\mathfrak{D}_1'$.*

There are four other systems of this type.

7. Connective relations. Now c_1, c_2, c_3, c_4, c_5, f form a 6-line. Hence, taking the next step in the Clifford and Morley chains, the circles, $\mathfrak{d}$, $\mathfrak{d}_1'$, $\mathfrak{d}_2'$, $\mathfrak{d}_3'$, $\mathfrak{d}_4'$, $\mathfrak{d}_5'$ pass through a point, p, the Clifford point of the 6-line.

Also, circles $\mathfrak{g}$, $\mathfrak{g}_1'$, $\mathfrak{g}_2'$, $\mathfrak{g}_3'$, $\mathfrak{g}_4'$, $\mathfrak{g}_5'$ pass through a point, q, the de Longchamps point of the 6-line.

And the centers of the circles $\mathfrak{d}$, $\mathfrak{d}_1'$, $\mathfrak{d}_2'$, $\mathfrak{d}_3'$, $\mathfrak{d}_4'$, $\mathfrak{d}_5'$ lie on a circle, the Morley circle of the 6-line.

It follows at once that:

(H) *The circles $\mathfrak{D}$, $\mathfrak{D}_1'$, $\mathfrak{D}_2'$, $\mathfrak{D}_3'$, $\mathfrak{D}_4'$, $\mathfrak{D}_5'$ pass through a point P.*
(I) *The circles $\mathfrak{G}$, $\mathfrak{G}_1'$, $\mathfrak{G}_2'$, $\mathfrak{G}_3'$, $\mathfrak{G}_4'$, $\mathfrak{G}_5'$ pass through a point, Q.*
(J) *The inverse points of K with respect to the circles $\mathfrak{D}$, $\mathfrak{D}_1'$, $\mathfrak{D}_2'$, $\mathfrak{D}_3'$, $\mathfrak{D}_4'$, $\mathfrak{D}_5'$ lie on a circle.*

8. The chain. In each of the six systems of circles described in section 6, there are six circles passing respectively through the points L, L_1, L_2, L_3, L_4, L_5. Thus, inverting and re-inverting with respect to circles with each of these points as center and with definite radii, we may obtain thirty-six new systems in each of which theorems like (A) to (G) apply, linked by six systems to which theorems like (H) to (J) apply. This process may be continued indefinitely, with 6^p new

systems obtained at the pth repetition of the process linked by 6^{p-1} of the other systems.

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ROTORS WITHIN ROTORS

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"As if a wheel had been in the midst of a wheel."—Ezekiel x.10

1. Introduction. This note describes a relationshp between two curves each of which is known for its own remarkable properties. One curve is an involute of the deltoid. The deltoid is known also as the tricuspid, or the hypocycloid of three cusps. The other curve is known as the curve of Ribaucour. It is one of the parallels to the astroid of the hypocycloid of four cusps. Hence, the curve of Ribacour is called a para-astroid [1, 2].

The following theorem will be demonstrated.

THEOREM. *The curve of Ribaucour can be inscribed in the deltoid involute and may be turned continously in a constrained manner through all orientations.*

2. The deltoid involute rotor in the square. The deltoid is traced by the ends of a line segment of length $4b$ whose midpoint moves along a fixed circle of radius b through the angle 2ϕ while the line segment rotates in the opposite direction about this moving midpoint through the angle ϕ. The well-known fact that the line segment is always tangent to the curve will be shown. The tangential polar equation of the deltoid is

$$p = b \sin 3\phi \tag{1}$$

where p is the distance from the origin to the tangent whose inclination with the x-axis is $90° - \phi$. The right member is called the supporting function (translating Minkowski's term *Stützfunktion*).

In the rectangular coordinate system, if (x_E, y_E) is the contact point, s_E is the element of arc length traced by E, and $p' = dp/d\phi$, we have in general

$$(2) \qquad \begin{aligned} x_E &= p\cos\phi - p'\sin\phi, & dx_E &= -\sin\phi\, ds_E \\ y_E &= -p\sin\phi - p'\cos\phi, & dy_E &= -\cos\phi\, ds_E, \end{aligned}$$

and the radius of curvature $\rho = ds/d\phi$ and the intercept n of the normal are given by the formulas

$$(3) \qquad \rho = s_E' = p + p'' \qquad s_E = p' + \int p\, d\phi$$
$$n = -y\csc\phi = p + p'\cot\phi.$$

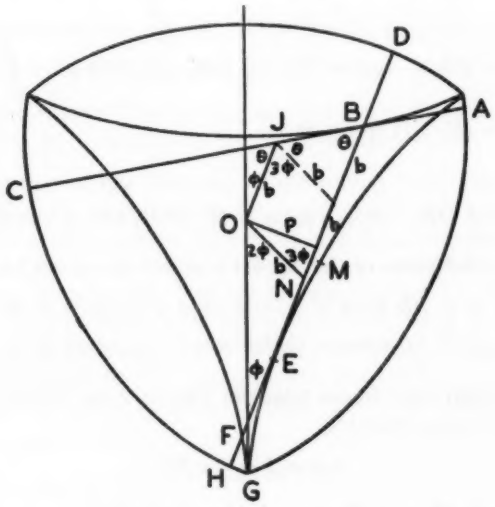

FIG. 1

From Figure 1, the familiar parametric coordinates of the variable point B on the curve are as follows:

$$(4) \qquad \begin{aligned} x_B &= 2b\sin\phi + b\sin 2\phi, & x_B' &= 4b\cos(3\phi/2)\cos(\phi/2) \\ y_B &= 2b\cos\phi - b\cos 2\phi, & y_B' &= 4b\cos(3\phi/2)\sin(\phi/2). \end{aligned}$$

We find that the arc length s_B, measured from the midpoint of an arch for a change of $\phi/2$ in the inclination of the tangent, is given by

$$s_B' = 4b\cos(3\phi/2), \qquad s_B = (8b/3)\sin(3\phi/2).$$

Also, since the slope at B is $y'/x' = \tan(\phi/2)$, the angle between the two tangents at B in Fig. 1 is $\theta = 90° - \phi - \phi/2$.

In equations (4), if ϕ is replaced by $180° - 2\phi$, the coordinates take the values in equations (2) showing that the contact point E is on the locus of point B.

If a straight line of length $16b/3$ (the length of one of the three arches of the deltoid) is placed so that it is tangent to a cusp, one end coinciding with a vertex, and then rolled over the deltoid without slipping, its ends will describe an involute of the deltoid which passes through all the vertices of the deltoid. Since this rolling line of fixed length is normal to the involute at its ends, the involute is a curve of constant width. If a pair of parallel tangents is held fixed, the involute may be turned between them, remaining tangent to both lines. Another pair of parallel tangents may be added without restraining the rotation. Therefore, the deltoid involute is said to be a rotor in a square (or any rhombus).

Since

$$DE = 16b/3 - GE = 8b/3 + EM = 8b/3 + p' - \int p\,d\phi$$

$$= (8b/3)(1 + \cos 3\phi)$$

and

$$BD = DE - p' + b\sin\phi - 2b = (2b/3)(1 + \cos 3\phi), \tag{5}$$

the parametric coordinates of the deltoid involute are given by

$$\begin{aligned} x_D &= x_B + BD\sin\phi = (b/3)(8\sin\phi + 2\sin 2\phi + \sin 4\phi) \\ y_D &= y_B + BD\cos\phi = (b/3)(8\cos\phi - 2\cos 2\phi + \cos 4\phi). \end{aligned} \tag{6}$$

3. The Ribaucour rotor in the triangle. The curve of Ribaucour is described by the tangential polar equation

$$p = a(3 + \cos 2\theta). \tag{7}$$

In parametric form, the coordinates of this curve are

$$\begin{aligned} x &= (a/2)(9\cos\theta - \cos 3\theta) = 2a(3\cos\theta - \cos^3\theta) \\ y &= (a/2)(3\sin\theta - \sin 3\theta) = 2a\sin^3\theta. \end{aligned} \tag{8}$$

The Cartesian form is

$$x^2 + y^2 + 3(2ay^2)^{2/3} = 16a^2. \tag{9}$$

This curve, which resembles an ellipse, has a major axis of twice the length of the minor axis. If an equilateral triangle is circumscribed about the curve, the sum of the distances of its sides from the origin is constant, as is shown in the following equation:

$$
\begin{aligned}
&p(\theta) + p(\theta + 120°) + p(\theta - 120°) \\
&\quad = a(3 + \cos 2\theta) + a(3 + \cos 2\theta \cos 120° - \sin 2\theta \sin 120°) \\
&\quad\quad + a(3 + \cos 2\theta \cos 120° + \sin 2\theta \sin 120°) = 9a.
\end{aligned} \tag{10}
$$

Since the altitude of the triangle is equal to the constant sum of these distances, the triangle has the same size for all orientations. Therefore, the curve of Ribaucour is a rotor in an equilateral triangle.

4. The normal of the curve of Ribaucour. The distance between the point of contact and the foot of the perpendicular from the origin to the tangent is given by p'. If the intercept of the normal between the curve and the major axis is denoted by n, its value is derivable as follows:

$$n = p + p' \cot\theta = a(3 + \cos 2\theta - 2 \sin 2\theta \cot\theta) = a(1 - \cos 2\theta). \tag{11}$$

The radius of curvature ρ is given by the following equation:

$$\rho = p + p'' = a(3 + \cos 2\theta - 4 \cos 2\theta) = a(3 - 3 \cos 2\theta) = 3n. \tag{12}$$

This relation (the radius of curvature is three times the normal intercept) is sometimes used as the definition of the curve of Ribaucour.

5. Demonstration of the theorem. If the rolling line of Section 2 carries a plane with it, the successive positions of the fixed deltoid in the moving plane will have a curve as an envelope. This curve is said to be molded in the moving plane by the fixed deltoid involute. In Figure 1, construct the line AC tangent to an arch of the deltoid at B. Through B pass another line DH tangent at E to another arch of the deltoid, cutting the deltoid again at F. Denote the midpoint of BF by N. Let the angle GON be 2ϕ, and the angle between BF and OG be ϕ, as described in Section 2. Similarly, the line OJ has rotated through the angle ϕ while the line AC has rotated through $\phi/2$ in the opposite direction. We have

$$2\theta - 3\phi = 180°$$

from which the angle θ between the two tangents through B is given by

$$\theta = 90° - 3\phi/2. \tag{13}$$

This confirms a result derived in Section 2.

The point B is the instantaneous center of rotation of the plane which carries AC. Therefore, the curve molded in this plane must touch the deltoid involute at D which is the foot of the perpendicular from B. From equations (5) and (13) we have:

$$BD = (2b/3)(1 + \cos 3\phi) = (2b/3)(1 - \cos 2\theta). \tag{14}$$

Comparison of this value of the normal intercept with equation (11) shows that

the molded curve is the curve of Ribaucour with $a=2b/3$.

In Figure 2, the curve of Ribaucour lies inside the deltoid involute. An equilateral triangle is circumscribed about the curve of Ribaucour and a square is circumscribed about the deltoid involute. The figure illustrates one of the ∞^3 configurations of the rotors and their polygons.

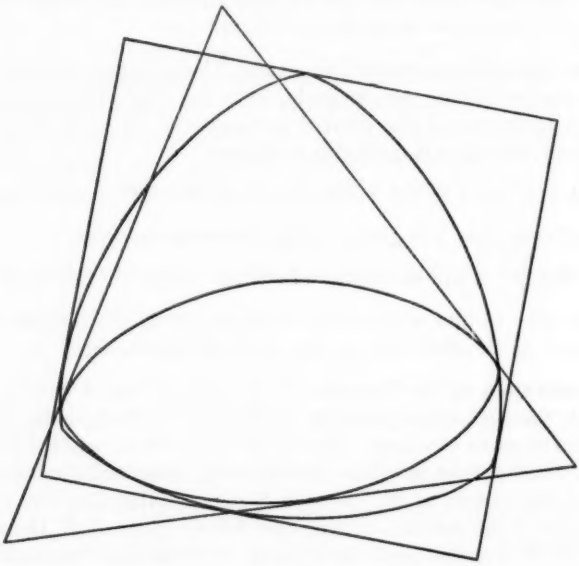

FIG. 2

6. Parallel rotors. The theorem can be generalized to parallel rotors. If the deltoid involute is replaced by a parallel curve at any arbitrary distance, the curve of Ribaucour can be replaced by a parallel curve separated by the same distance. These curves also are rotors in the square and the triangle respectively. As the arbitrary distance is increased, the curves approach circular shapes. Internal parallel curves cannot be used since the radii of curvature at the vertices of the original curves are zero; internal parallel curves would no longer be convex.

7. Similar theorems. The theorem derived here is similar to a chain of known theorems. Every hypocycloid of n cusps is a rotor in another hypocycloid of $n+1$ cusps. Also, every epicycloid of n cusps is a rotor in an epicycloid of $n+1$ cusps. Their parallel curves have been applied in the design of rotary pump elements [3].

The theorem described here is not derivable as a special case of the foregoing theorems. Similarly, the possible generalization to involutes of higher

order hypocycloids is not an immediate consequence.

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LINEAR RECURRENCE RELATIONS*

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1. Introduction. The relation

$$u_{n+1} = u_n + u_{n-1},$$

which is satisfied by the Fibonacci sequence $u_{-1}=0$, $u_0=1$, $u_1=1$, $u_2=2$, $u_3=3$, $u_4=5$, $\cdots$, can be generalized. For example, the sequence of Lucas [5] can be defined similarly from the relation

$$u_{n+1} = Pu_n - Qu_{n-1}, \tag{1}$$

where u_{-1}, u_0 have the same values as before, and u_1, u_2, $\cdots$ are calculated in succession from (1). Another linear recurrence relation is discussed in [6] in connection with a divisibility question. The most general linear recurrence relation (over integers) can be written in the form

$$u_{n+1} = \alpha_0 u_n + \alpha_1 u_{n-1} + \cdots + \alpha_{k-1} u_{n-k+1} + a, \tag{2}$$

where k is positive, a and α_i are integers ($i=0, \cdots, k-1$), and α_{k-1} is not 0. The integer k is called the degree of the relation, and is fixed henceforth. If (initial) integral values $u_{-1}, u_0, \cdots, u_{k-2}$ are prescribed, the values $u_{k-1}, u_k, \cdots$ can be computed in succession from (2). If α_{k-1} happens to be 1 or -1, the sequence u_k can be computed for negative values of k also, by solving (2) for u_{n-k+1}, as in [6]. Linear recurrence relations and the sequences associated with them are useful in solving certain problems concerning primality or divisibility; Lehmer [4] gave a test for the primality of Mersenne numbers 2^p-1 (p, a prime) which has been used to discover a prime with several hundred digits; this test was based on properties of the Lucas sequence. The relation $u_n \mid u_{nr}$ holds generally for Lucas sequences. An illuminating account of linear recurring sequences is given in [3].

Two interesting (known) properties of recurring sequences are studied in this article by the use of matrix methods. Firstly, the residues mod p of the u_n must be periodic; so that secondly, if u_{-1} is 0, then some u_n must be divisible by

* The writing of this article was sponsored by the Office of Ordnance Research, U. S. Army, under contract DA-04-200-ORD-113. This is technical report 7 under that contract.

p, if p is a prime not dividing α_{k-1}. General theorems about matrices lead quickly to fairly precise results concerning the size of this period. On the other hand, it would be possible to establish theorems concerning the period of a matrix belonging to a certain class of matrices by starting with results concerning recurrence relations which have already been established in other ways.

2. The matrix formula. Let A stand for the $(k+1)\times(k+1)$ matrix

$$\begin{pmatrix} 1, & 0, & 0, & 0, & \cdots & ,0, & a \\ 0, & 0, & 0, & 0, & \cdots & , 0, & \alpha_{k-1} \\ 0, & 1, & 0, & 0, & \cdots & , 0, & \alpha_{k-2} \\ 0, & 0, & 1, & 0, & \cdots & , 0, & \alpha_{k-3} \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0, & 0, & 0, & 0, & \cdots & , 1, & \alpha_0 \end{pmatrix}.$$

In case k is 1, A is the matrix $\left(\begin{smallmatrix} 1, & a \\ 0, & \alpha_0 \end{smallmatrix}\right)$. Let B stand for the $(k+2)\times(k+2)$ matrix

$$\begin{pmatrix} 0, & 1, & u_{-1}, & \cdots & , u_{k-2} \\ 0 & & & & \\ 0 & & A & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & & & & \end{pmatrix}.$$

From (2), the formula

$$B^{n+1} = \begin{pmatrix} 0, & 1, & u_{n-1}, & \cdots & , u_{n+k-2} \\ 0 & & & & \\ 0 & & A^{n+1} & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & & & & \end{pmatrix} \qquad (n > -1) \tag{3}$$

is easily proved by induction.

3. Periodicity. Apparition of primes. Since the vector $(u_{n-1}, u_n, \cdots, u_{n+k-2})$ can take at most p^k values which are distinct modulo the prime p, the following theorem is true.

THEOREM 1. (Lagrange) *Modulo an arbitrary prime, the values u_n given by the linear recurrence relation* (2) *are periodic with period not exceeding p^k.*

Not all periods less than $1+p^k$ are actually possible. To show this, it is convenient to use some information from the theory of automorphisms of abelian groups.

Let G be an elementary abelian group of order p^{k+1}, that is, an abelian group

generated by $k+1$ independent generators, each of order p. [G is isomorphic to the additive group of $(k+1)$-vectors $(\beta_0, \beta_1, \cdots, \beta_k)$, the components of the vectors being integers mod p.] An automorphism of G can be described by mapping the first generator $(1, 0, \cdots, 0)$ onto any of the $p^{k+1}-1$ elements which have order p; by mapping the second generator $(0, 1, \cdots, 0)$ onto any of the $p^{k+1}-p$ elements independent of this element; *etc.* Thus the order of the group of automorphisms of G is $(p^{k+1}-1)(p^{k+1}-p)\cdots(p^{k+1}-p^k)$, which is later denoted by $\phi(p, k)$.

COROLLARY. *The order of any automorphism of G is a divisor of $\phi(p, k)$.*

Actually, more than this is known [1].

THEOREM. (Burnside) *The group of automorphisms of G is isomorphic to the group of $(k+1)\times(k+1)$ nonsingular matrices* mod p [2, p. 242].

COROLLARY. *The order of any such matrix is a divisor of $\phi(p, k)$.*

Thus if A mod p is non-singular, its order is a divisor of this number. The condition for this is that α_{k-1} shall not be divisible by p. Here, the order of A mod p means the smallest positive exponent r such that the relation $A^r \equiv I$ (mod p) holds. When A mod p is non-singular, such an exponent must exist.

THEOREM. *Let r be the order of A* mod p. *Then the relation $B^{r+1} \equiv B$* (mod p) *holds.*

This theorem follows from the relation $B^{r+1} = BB^r$ together with the displayed matrix equation (3).

COROLLARY. *Modulo an arbitrary prime p which does not divide α_{k-1}, the values u_n given by the linear recurrence relation* (2) *are periodic with period dividing $(p^{k+1}-1)(p^{k+1}-p)\cdots(p^{k+1}-p^k)$; indeed the period divides the order r of A* mod p.

COROLLARY. *If any one of $u_{-1}, u_0, \cdots, u_{k-2}$ is* 0, *then every prime p, not a divisor of α_{k-1}, has a finite rank of apparition; that is, an infinite number of terms u_n are divisible by the fixed prime p.*

The methods of this article can be generalized in various directions, which will be noted by the sophisticated reader.

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MATHEMATICAL NOTES

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A NOTE ON WOLSTENHOLME'S THEOREM

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1. A well-known theorem of Wolstenholme asserts that if p is a prime greater than 3, then

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p-1}\equiv 0 \pmod{p^2}. \tag{1.1}$$

For a proof, see for example [2, p. 87]. It is perhaps of interest to examine the sum

$$S_{k,p}=\frac{1}{kp+1}+\frac{1}{kp+2}+\cdots+\frac{1}{kp+p-1}, \tag{1.2}$$

where k is an arbitrary integer.

Put

$$\prod_{r=1}^{p-1}(x-kp-r)=x^{p-1}-A_1^{(k)}x^{p-2}+\cdots+A_{p-1}^{(k)}. \tag{1.3}$$

If we take $x=(2k+1)p$ in (1.3), it is clear that

$$A_{p-2}^{(k)}-(2k+1)pA_{p-3}^{(k)}+(2k+1)^2p^2A_{p-4}^{(k)}-\cdots=0. \tag{1.4}$$

Since by (1.3)

$$A_r^{(k)}\equiv A_r^{(1)}\equiv 0 \pmod{p}, \qquad (1\leqq r\leqq p-2),$$

it is clear that $A_{p-2}^{k}\equiv 0 \pmod{p^2}$ for all k (provided $p>3$). Moreover if $p^e \mid (2k+1)$ then

$$A_{p-2}^{(k)}\equiv 0 \pmod{p^{e+2}}.$$

Also (1.4) implies the congruence

$$A_{p-2}^{(k)}\equiv (2k+1)pA_{p-3}^{(k)} \pmod{(2k+1)^2p^3}; \tag{1.5}$$

indeed (1.5) holds (mod $(2k+1)^2p^4$) for $p>5$.

Since

$$S_{k,p}=A_{p-2}^{(k)}/[(kp+1)(kp+2)\cdots(kp+p-1)],$$

it is clear in particular that

$$S_{k,p} \equiv 0 \pmod{p^{\alpha+2}}$$

provided $p^\alpha \mid (2k+1)$, $p>3$. As for $p=3$, we have

$$S_{k,3} = \frac{1}{3k+1} + \frac{1}{3k+2} = \frac{3(2k+1)}{(3k+1)(3k+2)},$$

so that $S_{k,3} \equiv 0 \pmod{p^{\alpha+1}}$. It therefore follows that for properly chosen k, $S_{k,p}$ is divisible by arbitrarily high powers of p.

2. Wolstenholme's theorem has been generalized by Leudesdorf. In the simplest case we have

$$S_m = \sum_{r=1,(r,m)=1}^{m} \frac{1}{r} \equiv 0 \pmod{m^2}$$

for $(m, 6)=1$; for a proof of this result see [2, p. 100], also [1]. Extending (1.2) we define

$$S_{k,m} = \sum_{r=1}^{m}{}' \frac{1}{km+r}, \tag{2.1}$$

where the prime denotes that the summation is restricted to r prime to m, and k is an arbitrary integer.

Exactly as in [1], we have

$$2S_{k,m} = \sum_{r=1}^{m}{}' \left(\frac{1}{km+r} + \frac{1}{km+m-r} \right) = (2k+1)m \sum_{r=1}^{m}{}' \frac{1}{(km+r)(km+m-r)}$$

$$= -(2k+1)m^2 \sum_{r=1}^{m}{}' \frac{1}{(km+r)^2(km+m-r)} + (2k+1)m \sum_{r=1}^{m}{}' \frac{1}{(km+r)^2}.$$

Now

$$\sum_{r=1}^{m}{}' \frac{1}{(km+r)^2} \equiv \sum_{r=1}^{m}{}' \frac{1}{r^2} \equiv 0 \pmod{m}. \tag{2.2}$$

Chowla derives the latter congruence from

$$\sum{}' \frac{1}{r^2} \equiv \sum{}' \frac{1}{(ar)^2} \pmod{m},$$

where a is chosen so that $a^2 \not\equiv 1 \pmod{p}$ for any prime p dividing m; since m is prime to 6 it is possible to find such a. Alternatively the result can be proved by means of the formula

$$\sum_{r=1}^{m}{}' \frac{1}{r^2} \equiv \phi\left(\frac{m}{p^\alpha}\right) \sum_{r=1,p\nmid r}^{p^\alpha} \frac{1}{r^2} \pmod{p^\alpha},$$

where $p^\alpha \mid m$, $p^{\alpha+1} \nmid m$.

Combining (2.1) and (2.2) we see that

$$S_{k,m} \equiv 0 \pmod{(2k+1)m^2} \qquad ((m, 6) = 1). \tag{2.3}$$

In particular for properly chosen k, it is clear that $S_{k,m}$ is divisible by arbitrarily high powers of m; this last is true for all m.

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ON RULED SURFACES

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1. Introduction. Treatments of skew curves invariably deal with the theory of the developable surfaces in space generated by the planes of the moving trihedral. The following represents an attempt to discuss the ruled surface in space generated by the axis of rotation of the rigid body supporting the trihedral throughout its progress and moving with it. For the sake of simplicity, we shall identify s, the distance covered by the moving point as it progresses along the curve, with t, the time required.

2. The ruled surface. Let (x, y, z) be the coordinates, relative to fixed axes in space, of the moving point on the curve associated with the value s, and let the direction cosines of the tangent, principal normal, binormal, be

$$(l_1, m_1, n_1), \qquad (l_2, m_2, n_2), \qquad (l_3, m_3, n_3) \tag{1}$$

respectively. Let (u, v, w) be the coordinates, relative to the trihedral at (x, y, z) of a given point in the rigid body, and let (X, Y, Z) be the coordinates of the same point, relative to the fixed axes in space. We have, therefore,

$$X = x + l_1u + l_2v + l_3w, \tag{2}$$

with corresponding expressions for Y and Z.

Employing κ and τ to denote the curvature and torsion of the curve at (x, y, z) and making use of the Frenet-Serret formulae, we have

$$\dot{X} = l_1(1 - v\kappa) + l_2(u\kappa + w\tau) - l_3(v\tau), \tag{3}$$

with corresponding expressions for $\dot{Y}$ and $\dot{Z}$, where the dot notation indicates derivation with respect to s at (x, y, z). The square of the velocity of the moving point (X, Y, Z) is, therefore,

$$(1 - v\kappa)^2 + (u\kappa + w\tau)^2 + (v\tau)^2, \tag{4}$$

which is at its least on the axis of rotation

$$u\kappa + w\tau = 0, \qquad v = \frac{\kappa}{R^2} \tag{5}$$

in the rigid body, where R^2 stands for $\kappa^2+\tau^2$. This line is the s-generator of the ruled surface in space, and its equations become

$$\begin{gathered}(X - x)(l_1\kappa + l_3\tau) + (Y - y)(m_1\kappa + m_3\tau) + (Z - z)(n_1\kappa + n_3\tau) = 0, \\ (X - x)l_2 + (Y - y)m_2 + (Z - z)n_2 = \frac{\kappa}{R^2},\end{gathered} \tag{6}$$

when made to refer to the fixed axes in space.

The square of the velocity, as given in (4), may be written in the form

$$T^2 + \Delta^2 R^2, \tag{7}$$

where T denotes τ/R, the velocity in the direction of the s-generator, while Δ^2 denotes

$$\left(\frac{u\kappa + w\tau}{R}\right)^2 + \left(v - \frac{\kappa}{R^2}\right)^2,$$

the square of the distance of the moving point from the axis of rotation, so that ΔR represents the velocity of rotation and R the angular velocity of rotation about this line. In particular, for the point (x, y, z) itself, expression (7) reduces to unity, as required. If s is allowed to vary, equations (6) supply the ruled surface in space.

3. The curve of striction. The s-generator, as given in (6), has direction cosines

$$\left(\frac{l_1\tau - l_3\kappa}{R}, \frac{m_1\tau - m_3\kappa}{R}, \frac{n_1\tau - n_3\kappa}{R}\right). \tag{8}$$

To these we must add

$$\frac{\kappa\dot{\tau} - \tau\dot{\kappa}}{R^2}\,ds \text{ times } \left(\frac{l_1\kappa + l_3\tau}{R}, \frac{m_1\kappa + m_3\tau}{R}, \frac{n_1\kappa + n_3\tau}{R}\right)$$

respectively, to obtain the direction cosines of the $(s+ds)$-generator. It follows that the common perpendicular to these two generators has direction cosines

$$(l_2, m_2, n_2), \tag{9}$$

which means that it is parallel to the principal normal at (x, y, z).

But the principal normal at (x, y, z) intersects the s-generator in the point (X, Y, Z) given by

$$\frac{X - x}{l_2} = \frac{Y - y}{m_2} = \frac{Z - z}{n_2} = \frac{\kappa}{R^2}. \tag{10}$$

It is easy to see that (X, Y, Z) is the point of striction on the s-generator. Indeed,

$$dX = \left(\frac{l_1\tau - l_3\kappa}{R}\right)Tds + l_2 d\,\frac{\kappa}{R^2}, \tag{11}$$

with corresponding expressions for dY and dZ. In other words,

$$X + l_2 d\,\frac{\kappa}{R^2} = X + dX - \left(\frac{l_1\tau - l_3\kappa}{R}\right)Tds,$$

with corresponding expressions for $Y+m_2 d(\kappa/R^2)$ and $Z+n_2 d(\kappa/R^2)$. That is, to proceed from (X, Y, Z) a distance of $d(\kappa/R^2)$ in the direction indicated in (9) lands us finally at the same point as to proceed from $(X+dX, Y+dY, Z+dZ)$ a distance of $T\,ds$ in the direction opposite to that indicated in (8), and this means that (X, Y, Z) is the point of striction on the s-generator. Strictly speaking, we should have used the direction of the $(s+ds)$-generator rather than that of the s-generator, but the conclusion would still have been the same, since none but the coefficients of higher powers of ds than the first could have been affected. If s is allowed to vary, equations (10) supply the curve of striction.

4. Properties. The curve of striction in space corresponds to one or more segments of the principal normal in the rigid body, with the possibility of such segment or segments being swept out again and again. The distance between the s-generator and the $(s+ds)$-generator in space is $d(\kappa/R^2)$, which is also the distance between them in the rigid body. The points of striction on these two generators in space are advanced, one relative to the other, a distance of Tds in their virtually common direction as given in (8), whereas in the rigid body there is no such advance whatever of one relative to the other, since both points lie on the principal normal.

The tangent plane to the ruled surface at the point of striction on the s-generator passes through

(i) the s-generator, with direction cosines as given in (8),

(ii) the principal normal orthogonal to (i), with direction cosines as given in (9),

(iii) the tangent line to the curve of striction at the point, with direction ratios (dX, dY, dZ) as begun in (11) and completed immediately after.

Where θ and ϕ denote the angles which (iii) makes with (i) and (ii) respectively, we easily find that

$$\frac{\cos\theta}{Tds} = \frac{\cos\phi}{d\,\dfrac{\kappa}{R^2}} = \frac{1}{\left\{(Tds)^2 + \left(d\,\dfrac{\kappa}{R^2}\right)^2\right\}^{1/2}}.$$

The s-generator is tangent to the curve of striction at (X, Y, Z) as given in (10) when and only when κ/R^2 is constant. In that case, and provided the orig-

inal curve is not a cylindrical helix, the ruled surface is the tangent surface of the curve of striction and is developed by its osculating planes, so that the binormal of the curve of striction is identical with the principal normal of the original curve. The proof of this is routine. If however κ/R^2 is constant, with the original curve being a cylindrical helix, it is clear that κ and τ are both constant, which means that the osculating planes of the curve of striction prove to be indeterminate. Also, the tangent plane to the ruled surface remains the same for all the points of contact on a single generator, provided κ/R^2 is constant, but rotates about the generator as the point of contact recedes along it, provided κ/R^2 is not constant.

In particular, if the original curve is not skew, the ruled surface is a cylinder, with generators normal to the plane of the original curve. In that case, the curve of striction is not unique but may be taken as the intersection of the cylinder with the plane of the original curve and so is the evolute of the original curve. In the special case where the original curve is a circle, the ruled surface is a single line normal to its plane and passing through the centre of the circle.

THE NON-EXISTENCE OF RATIONAL SOLUTIONS FOR CERTAIN DIFFERENCE EQUATIONS

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1. Introduction. In this paper we prove:

Theorem I. *The difference equation*

$$\frac{1}{x^k} = F(x+1) - F(x), \qquad k = 1, 2, \cdots, \tag{1}$$

has no rational solution with coefficients in the complex field.

Theorem II. *The difference equation*

$$\frac{1}{x^k} = F(x+1) + F(x), \qquad k = 1, 2, \cdots, \tag{2}$$

has no rational solution with coefficients in the complex field.

We present two methods of proof.

2. Theorem I. $k=1$. Suppose that $F(x)$ is a rational function of x. Then we can write

$$F(x) = \frac{R(x)}{T(x)} \tag{3}$$

where $R(x)$ and $T(x)$ are polynomials in x having no factor in common and of degree r and t respectively. Hence

$$\frac{1}{x} = \frac{R(x+1)}{T(x+1)} - \frac{R(x)}{T(x)}. \tag{4}$$

Let x take successively the values $1, 2, \cdots, n$. (We observe that $T(x)$ vanishes for none of these values of x for were it to vanish for one such value it would vanish for infinitely many.) Adding the resulting equations we have

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} = \frac{R(n+1)}{T(n+1)} - \frac{R(1)}{T(1)}. \tag{5}$$

Letting $n \to \infty$ we consider two cases.

Case 1. $r \leqq t$. The left side of (5) grows without bound while the right side approaches a constant.

Case 2. $r > t$. Both sides of (5) grow large but with different orders of magnitude, the left side on the order of $\log n$, the right side as a power of n.

Thus the assumption that $F(x)$ is rational leads to a contradiction.

3. Theorem I. $k > 1$. The above procedure lends itself to the more general case. Assuming first that k is an even integer, say $k = 2p$, $p \geqq 1$, and that $F(x)$ is rational, we have

$$\frac{1}{x^{2p}} = \frac{R(x+1)}{T(x+1)} - \frac{R(x)}{T(x)}. \tag{6}$$

Letting x take on the values $1, 2, \cdots, n$ and adding, there results

$$1 + \frac{1}{2^{2p}} + \cdots + \frac{1}{n^{2p}} = \frac{R(n+1)}{T(n+1)} - \frac{R(1)}{T(1)}. \tag{7}$$

As $n \to \infty$, the left side of (7) has as its limiting value the irrational number [1]

$$\frac{(2\pi)^{2p}}{2(2p)!} B_p \tag{8}$$

where B_p is a Bernoulli number. If the right side of (7) is to converge it has for its limit either zero or a number dependent upon the coefficients of $R(x)$ and $T(x)$. That this number is rational follows from (7) and from the fact that if a rational function has rational values for all positive integral values of x then its coefficients are rational numbers [2]. Therefore we are led once again to a contradiction.

There remains to consider the case where k is odd, say $k = 2p-1$, $p \geqq 1$. Writing

$$\frac{1}{x^{2p-1}} = F(x+1) - F(x), \tag{9}$$

and taking derivatives we have

$$\frac{1}{x^{2p}} = \frac{F'(x+1) - F'(x)}{-2p+1}. \tag{10}$$

But the rationality of $F(x)$ implies the rationality of $F'(x)$ and (10) is of the same form as (6). Thus we have an immediate contradiction.

4. Theorem II. In a similar fashion we examine equation (2). Again assuming $F(x)$ a rational function and considering first $k=2p$, $p\geqq 1$, we employ the method of §3, this time alternately adding and subtracting the equations that result from letting x take successively the values $1, 2, \cdots, n$. We obtain

$$1 - \frac{1}{2^{2p}} + \cdots + \frac{(-1)^{n+1}}{n^{2p}} = (-1)^{n+1}F(n+1) + F(1). \tag{11}$$

The limit of the left side as $n\to\infty$ is the irrational number [1]

$$\frac{(2^{2p-1}-1)\pi^{2p}}{(2p)!}B_p. \tag{12}$$

Since the right side of (11) must have, employing the argument of §3, a rational number as its limiting value, it follows that the assumption that $F(x)$ is a rational function is not true.

For the case $k=2p-1$, $p\geqq 1$, we need only to employ differentiation as in §3 to reach a contradiction.

5. Alternative proof. Another method of proof for which I am indebted to L. Carlitz is as follows. As before, with respect to (1), let

$$\frac{1}{x^k} = \frac{R(x+1)}{T(x+1)} - \frac{R(x)}{T(x)}. \tag{13}$$

Clearing of fractions we have

$$T(x)T(x+1) = x^kR(x+1)T(x) - x^kR(x)T(x+1). \tag{14}$$

Since $R(x)$ and $T(x)$ have no factor in common, (14) implies on the one hand that $T(x)$ divides $x^kT(x+1)$, on the other that $T(x+1)$ divides $x^kT(x)$. Thus we have

$$x^kT(x+1) = T(x)M(x), \qquad x^kT(x) = T(x+1)N(x), \tag{15}$$

where $M(x)$ and $N(x)$ are polynomials. Hence $x^{2k}=M(x)N(x)$ and so $M(x)=N(x)=\pm x^k$ so that $T(x+1)=\pm T(x)$ which is evidently impossible.

For equation (2) the proof is exactly the same with a change of sign in (13) and (14).

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MATRICES AND POLYNOMIALS

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1. Introduction. In 1932, Pierce [1] gave a method for computing the resultant of two polynomials by applying a theorem of Frobenius to the companion matrices of the polynomials. The writer [2] showed how the degree of the greatest common divisor is determined from this form of the resultant. In the present note it is shown how the greatest common divisor may be determined from this form of the resultant.

2. The greatest common divisor of two polynomials. Let $c=(c_1, c_2, \cdots, c_n)$ be a vector in n-space and denote by $c(x)$ the polynomial $c_1+c_2x+\cdots+c_nx^{n-1}$. This correspondence between vectors of n-space and polynomials of virtual degree $n-1$ will be assumed throughout this paper. Basic vectors e_i are defined by $e_i(x)=x^{i-1}$ so that $c=\sum_{i=1}^{n} c_ie_i$. Let C be the n-rowed matrix whose rows are $e_2, e_3, \cdots, e_n, c$. Then $e_iC=e_{i+1}$ $(i=1, 2, \cdots, n-1)$ and $e_nC=c$. If C^0 is defined to be the identity matrix, it follows that the rows of C^{i-1} $(i=1, 2, \cdots, n)$ are $e_i, e_iC, e_iC^2, \cdots, e_iC^{n-1}$ and the rows of C^n are $c, cC, cC^2, \cdots, cC^{n-1}$; hence $C^n=c(C)$. Also, if k is any other vector of n-space and $k(x)$ is the corresponding polynomial, the matrix $k(C)$ has the rows $k, kC, kC^2, \cdots, kC^{n-1}$.

THEOREM 1. *If k is a vector of n-space, $k(C)=0$ if, and only if, $k=0$.*

COROLLARY. *The minimum function of the matrix C is $f(x)=x^n-c(x)$.*

The matrix C is usually referred to as the companion matrix of the polynomial $f(x)$. If $h(x)=q(x)f(x)+r(x)$ then $h(C)=r(C)$ and the greatest common divisor of $h(x)$ and $f(x)$ is the greatest common divisor of $r(x)$ and $f(x)$. Hence, it is not necessary to consider polynomials of degree greater than $n-1$.

If $h(x)$ and $k(x)$ are two polynomials of degrees less than n, the first row of the matrix $h(C)k(C)$ is $hk(C)$ and the first row of $k(C)h(C)$ is $kh(C)$. But $h(C)k(C)=k(C)h(C)$; hence

THEOREM 2. *If h and k are two vectors of n-space, the vectors $hk(C)$ and $kh(C)$ are identical.*

If the vectors $k, kC, \cdots, kC^{r-1}$ are linearly independent and $kC^r=k(h_1I+h_2C+\cdots+h_rC^{r-1})$ then $kC^s(s>r)$ is also a linear combination of the first r rows of $k(C)$. That is, the first r rows of $k(C)$ form a basis for the row space of $k(C)$ and consequently $k(C)$ is of rank r. Thus we have

THEOREM 3. *If the matrix $k(C)$ is of rank r the first r rows of $k(C)$ are linearly independent.*

We note that $kC^r=kh(C)$ and indicate by $g(x)$ the polynomial $x^r-h(x)$ where $h(x)$ is of degree $r-1$. Then $kg(C)=0$ and $g(x)$ is the minimum function of C relative to the vector k. But $kg(C)=0$ if, and only if, $k(C)g(C)=0$ and the latter

is true if, and only if, $f(x)$ is a factor of $k(x)g(x)$. Since $g(x)$ is the monic polynomial of lowest degree such that $k(x)g(x)$ is divisible by $f(x)$ it follows that $g(x) = f(x)/d(x)$ where $d(x)$ is the greatest common divisor of $f(x)$ and $k(x)$. This establishes

THEOREM 4. *The degree of the greatest common divisor of $f(x)$ and $k(x)$ is the nullity of $k(C)$* [2].

Also from the above we have

THEOREM 5. *Let $k(x)$ be any polynomial and let $d(x)$ be the greatest common divisor of $k(x)$ and $f(x)$. If $k(C)$ is nonsingular, then $d(x)=1$. If $k(C)$ is of rank $r<n$ and $g(x)$ is the monic polynomial of degree r such that $gk(C)=0$ then $d(x) = f(x)/g(x)$.*

3. The matrix equation $AX=XB$. Consider now the matrix equation $CX=XD$ where D has m rows. The rows of CX are e_1X, e_1XD, e_1XD^2, $\cdots$, e_1XD^{n-1}; hence $e_1X=u_1D$ where u_1 is the first row of X and consequently the rows of X are u_1, u_1D, $\cdots$, u_1D^{n-1}. Also the last row of CX is $u_1c(D)$ and the last row of XD is u_1D^n so that $u_1f(D)=0$. In particular if $m=n$ and D is the companion matrix of $k(x)$, then X is a polynomial in D. If P is a nonsingular matrix and $A=PCP^I$ and $B=PDP^I$, the equation $AX=XB$ has the same solution as $CP^IXP=P^IXPD$. Hence, we have

THEOREM 6. *If A and B are nonderogatory matrices which can be transformed into rational canonical form by the same nonsingular matrix P the only solutions of $AX=XB$ are polynomials in B.*

If $f(x)$ and $k(x)$ are monic polynomials of degree n and their respective companion matrices are C and D, then $CX=XD$ if and only if $X=h(D)$ where $h(x)$ is a polynomial of degree less than n such that $hf(D)=0$. We have seen above that $hf(D)=0$ if and only if $h(D)f(D)=0$ and the latter is true if and only if $k(x)$ divides $h(x)f(x)$. We may then say that all solutions of $CX=XD$ are given by $X=h(D)$ where $h(x)$ is any polynomial of degree less than n having $k(x)/d(x)$ as a factor.

If A and B are as defined in Theorem 6, then $AX=XB$ is equivalent to $CP^IXP=P^IXPD$. Hence $P^IXP=h(D)$ and $X=Ph(D)P^I=h(B)$, where $h(x)$ is as described above. We also have the well known

COROLLARY. *If A is nonderogatory, X is commutative with A if, and only if, X is a polynomial in A.* [3]

References

1. T. A. Pierce, The practical evaluation of resultants, this MONTHLY, vol. 39, 1932, p. 161.

2. W. V. Parker, The degree of the highest common factor of two polynomials, this MONTHLY, vol. 42, 1935, p. 164. See also MacDuffee, this MONTHLY, vol. 57, 1950, p. 156 and [3] p. 45.

3. W. V. Parker, The matrix equation $AX=XB$, Duke Math. Journal, vol. 17, 1950, pp. 43–51.

NOTE ON POSITIVE POLYNOMIALS

S. S. ABHYANKAR, Harvard University

Introduction.* Increased attention has been given, during recent years, to the synthesis of electric networks. By synthesis is meant the design of an electric network which has certain properties. The impedance function of a network may be represented by a polynomial $P(X)$ whose coefficients are sums of products of the parameters of the network and whose independent variable X is the frequency.

If $P(X)$ is the impedance function of the network and $P_1(X)$, $P_2(X)$, $\cdots$, $P_m(X)$ are impedance functions of the various branches of the network then $P(X)=P_1(X)+P_2(X)+\cdots+P_k(X)$.

Obviously the smaller the number of branches the less costly the network. Hence it is the object of the engineer to synthesize a network with as few branches as possible. Certain physical requirements further demand that the roots of $P_i(X)$ be negative. Hence the problem is to determine the minimum number of polynomials whose coefficients are positive, whose roots are negative, and whose sum is equal to a given polynomial with positive coefficients. In this paper we shall determine this minimum number and also give a construction to compute these polynomials. It being equally simple, we shall deal with polynomials in any arbitrary ordered field. By a positive polynomial we shall mean a polynomial all of whose coefficients are positive numbers in the given ordered field. Now we state the

THEOREM. *Let $m(n)$ be the least integer such that any nth degree positive polynomial could be expressed as a linear combination with positive coefficients of not more than $m(n)$ positive polynomials each of which has all its roots negative.*

Then $m(n)$ = the least integer greater than $n/2$.

Proof. Without loss of generality we can assume that all the polynomials are monic.

Sufficiency.

Let $P(X)=X^n+a_1X^{n-1}+\cdots+a_n$ be any polynomial with $a_i>0$.

Case 1. n odd. Let $n=2p+1$. For $p=0$ the result is obvious. We apply induction in the case $p>0$. Assume it is true for $p-1$. Let a be the minimum of $a_2, a_3, \cdots, a_n$. Let ϵ be the minimum of

$$1, \frac{a}{n!2}, \frac{a}{n!2a_1}, \frac{a_1}{2}.$$

Let $\delta=\epsilon/(n-1)$. Let $Q(X)=(X+\delta)^{n-1}\ (X+a_1-\epsilon)=X^n+b_1X^{n-1}+\cdots+b_n$.

* I wish to thank Mr. H. H. Sun of Cornell University for help in the preparation of this introduction.

Then δ and $a_1-\epsilon$ are both positive. Also $b_1=a_1$ and for all $i>1$ we have $b_i<a_i$.* Therefore $P(X)=Q(X)+cP_1(X)$, where $c>0$, and $P_1(X)$ is a $[2(p-1)+1]$st degree monic positive polynomial.

Hence by assumption

$$P_1(X) = \sum_{i=1}^{p} c_iQ_i(X), \qquad \text{where } c_i > 0,$$

and $Q_i(X)$ are monic polynomials with all roots negative. Therefore

$$P(X) = Q(X) + \sum_{i=1}^{p} cc_iQ_i(X),$$

and the induction is complete.

Case 2. n even. The proof will be similar to that of Case 1 if we prove it for $n=2$. In this case, let ϵ be the minimum of 1, $a_1/3$, $a_2/2$. Let $c=a_1-2\epsilon$ and let $b=(a_2-\epsilon^2)/c$. Then ϵ, c and b are all positive and hence $P(X)=(X+\epsilon)^2+c(X+b)$ is a required decomposition.

Necessity.

For any given n the following nth degree positive polynomial $P_n(X)$ cannot be expressed as a positive linear combination of less than $\bar{n}+1$ positive polynomials with all roots negative, where $\bar{n}+1$ is the least integer greater than $n/2$.

$$P_n(X) = X^n + X^{n-1} + \left\{\binom{n}{2}n^2 + 1\right\} X^{n-2} + X^{n-3} + \left[\binom{n}{2}\left\{n^4 + a_2n^2\binom{n}{2}^2\right\} + 1\right] X^{n-4} + \cdots$$

where a_i, the coefficient of X^{n-i}, is a positive number determined, after a_1, a_2, $\cdots$, a_{i-1} are determined, in the following manner.

$$a_i = \begin{cases} 1 & \text{if } i \text{ is odd or zero.} \\ \left[\binom{n}{i}\sum_{j=0}^{k-1} a_{2j}\left\{n\binom{n}{2j}\right\}^{i-2j}\right] + 1 & \text{if } i = 2k > 0. \end{cases}$$

We now proceed to prove the impossibility of decomposing $P_n(X)$.

For $i=2k$ let $f_i^i=\binom{n}{i}a_{2j}\left\{n\binom{n}{2j}\right\}^{i-2j}$ if $i<k$ and let $f_i^k=1$. Also let $\phi_i^k=\sum_{j=p}^{k} f_i^j$. Then $a_i=\phi_i^0$.†

Let $P_n(X)=\sum_{l=1}^{s} c_lQ_l(X)$, with $c_l>0$ and Q_l a monic polynomial with all

* The proof of this inequality is as follows:—

$$b_i = \delta^{i-1}(a_1-\epsilon)\binom{n-1}{i-1} + \delta^i\binom{n-1}{i} \leqq \delta^{i-1}n!(a_1-\epsilon+\delta) \leqq n!\delta a_1 \leqq n!\epsilon a_1$$

$$\leqq n!\,\frac{a}{n!2a_1}\,a_1 = \frac{a}{2} \leqq \frac{a_i}{2} < a_i.$$

† The super indices of f, ϕ, g and d do not represent powers but are merely suffixes.

negative roots, be any decomposition of the required type with $s \leqq n$. Let $Q_l(X) = \sum_{i=0}^{n} g_i^l X^{n-i}$. We shall prove by induction:

(I) For all $p \leqq \bar{n}$ it is possible to choose p polynomials $Q_{l_1}, \cdots, Q_{l_p}$ such that if we let $R_p(X) = P_n(X) - \sum_{j=1}^{p} c_{l_j} Q_{l_j}(X) = \sum_{i=0}^{n} d_i^p X^{n-i}$ then $d_i^p \geqq \phi_i^p$ for all even $i \geqq 2p$.

For $p=0$ we have $R_0(X) = P_n(X)$ and hence (I) is obvious. Assume (I) to be true for some $q < \bar{n}$. By rearranging the subscripts of the terms $c_l Q_l$ we may assume that $R_p(X) = \sum_{l=q+1}^{s} c_l Q_l(X)$. Then $\sum_{l=q+1}^{s} c_l g_{2q}^l = d_{2q}^q \geqq \phi_{2q}^l = 1$. Therefore for some l between $q+1$ and s, which by rearranging the terms of $R_q(X)$ we may take as $l = q+1$, we must have $c_l g_{2q}^l \geqq 1/(s-q) \geqq 1/nk$. From now on we denote c_{q+1} by k. Then

$$g_{2q}^{q+1} \geqq \frac{1}{nk}. \tag{II}$$

Also $k g_{2q+1}^{q+1} \leqq a_{2q+1} = 1$. Therefore

$$g_{2q+1}^{q+1} \leqq \frac{1}{k}. \tag{III}$$

Then, if t is the degree of Q_{q+1}, $n \geqq t \geqq n-2q$. Let $x_1, \cdots, x_t$ be the roots of $Q_{q+1}(X) = 0$. Let $h = 2q+t-n$. Then $h \geqq 0$. Now we divide the argument into two cases.

Case 1. Let $h > 0$. Then $\sum x_{i_1} \cdots x_{i_h} = g_{2q}^{q+1} \geqq 1/nk$. Therefore at least one $x_{v_1} \cdots x_{v_h} \geqq 1/(nk\binom{t}{h})$, say $x_1 x_2 \cdots x_h$. Then for all $u > h$ we have $x_1 x_2 \cdots x_h x_u \leqq \sum x_{i_1} \cdots x_{i_{h+1}} = g_{2q+1}^{q+1} \leqq 1/k$. Therefore for all $u > h$ we have $x_u \leqq n\binom{t}{h} \leqq n\binom{n}{2q}$. But for all $v_1, \cdots, v_h$ we have $x_{v_1} \cdots x_{v_h} \leqq \sum x_{i_1} \cdots x_{i_h} = g_{2q}^{q+1} \leqq a_{2q}/k$. Therefore for all $j > 0$ and for all $v_1, \cdots, v_{h+2j}$ we have

$$x_{v_1} \cdots x_{v_{h+2j}} \leqq \frac{a_{2q}}{k} \left\{ n \binom{n}{2q} \right\}^{2j}.$$

Case 2. Let $h = 0$. Then $g_{2q}^{q+1} = 1$. Hence by (II) we have $1/k \leqq n$. Therefore by (III) we have $g_{2q+1}^{q+1} \leqq n$. Then for all j we have $x_j \leqq \sum x_i = g_{2q}^{q+1} \leqq n$. Hence for all $j > 0$ and for all $v_1, \cdots, v_{h+2j}$ we have $x_{v_1} \cdots x_{v_{h+2j}} \leqq n^{2j}$. But $k = k g_{2q}^{q+1} \leqq a_{2q}$. Therefore

$$n^{2j} \leqq \frac{a_{2q}}{k} \left\{ n \binom{n}{2q} \right\}^{2j}.$$

Thus in both the cases, for all $j > 0$ and for all $v_1, \cdots, v_{h+2j}$ we have $x_{v_1} \cdots x_{v_{h+2j}} \leqq a_{2q} \left\{ n\binom{n}{2q} \right\}^{2j}/k$. Therefore for all $j > 0$ we have

$$k g_{2q+2j}^{q+1} = k \sum x_{i_1} \cdots x_{i_{h+2j}} \leqq k \binom{t}{2q+2j} \frac{a_{2q}}{k} \left\{ n \binom{n}{2q} \right\}^{2j}$$

$$\leqq \binom{n}{2q+2j} a_{2q} \left\{ n \binom{n}{2q} \right\}^{2j} = f_{2q+2j}^{q}.$$

But $kg_{2q+2j}^{q+1}+d_{2q+2j}^{q+1}=d_{2q+2j}^{q}\geqq\phi_{2q+2j}^{q}$. Therefore for all even $i\geqq q+1$ we have $d_i^{q+1}\geqq\phi_i^q-f_i^q=\phi_i^{q+1}$.

Thus the induction is complete. Hence after choosing $\bar{n}$ polynomials Q_l, which again by rearranging we can take as $Q_1, Q_2, \cdots, Q_{\bar{n}}$, we are left with $R_{\bar{n}}(X)=\sum_{i=\bar{n}+1}^{s} c_iQ_i(X)$ for which $d_{2n}^{\bar{n}}\geqq 1$; so that $R_{\bar{n}}(X)\not\equiv 0$ and hence $s\geqq n+1$.

ABRIDGED SERIES FOR NUMERICAL EVALUATION

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If the sequence a_n decreases strictly to zero, then $s_m=\sum_1^m (-1)^{n-1}a_n\to S=\sum_1^\infty (-1)^{n-1}a_n$. The numerical evaluation of S can be simplified in certain cases.

Theorem. *If there exist numbers* Δ_n $(n=1, 2, 3, \cdots)$ *such that*

$$a_{2n}-a_{2n+1}\geqq\Delta_n\left|a_n-a_{2n+1}-a_{2n-1}\right|$$

then, for $n=1, 2, 3, \cdots$,

$$s_{2n+1}-S\geqq\delta_n\left|\sigma_n-S\right|$$

where $\delta_n=\operatorname{glb}_{m>n}\Delta_m$ *and* $\sigma_n=s_n+(-1)^n a_{2n+1}$.

Proof.

$$\begin{aligned}
s_{2n+1}-S&=\sum_{m=n+1}^{\infty}(a_{2m}-a_{2m+1})\\
&\geqq\sum_{m=n+1}^{\infty}\Delta_m\left|a_m-a_{2m+1}-a_{2m-1}\right|\\
&\geqq\sum_{m=n+1}^{\infty}\Delta_m\left|\sigma_m-\sigma_{m-1}\right|\\
&\geqq\delta_n\sum_{m=n+1}^{\infty}\left|\sigma_m-\sigma_{m-1}\right|\\
&\geqq\delta_n\left|\sigma_n-S\right|.
\end{aligned}$$

Example. Let $a_n=1/n$ in the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$. The hypothesis is satisfied if $\Delta_n<\frac{1}{2}(2n-1)$, and since Δ_n is increasing $\delta_n=\Delta_{n+1}=n+1/2$. Thus, for $n=49$,

$$\left|\sigma_{49}-S\right|\leqq\frac{2}{99}(s_{99}-S)<\frac{2}{99}(.01)<.00021.$$

Another familiar series to which this method is applicable is the Gregory series.

CLASSROOM NOTES

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ON A GRAPHICAL SOLUTION OF A CLASS OF DIFFERENTIAL EQUATIONS

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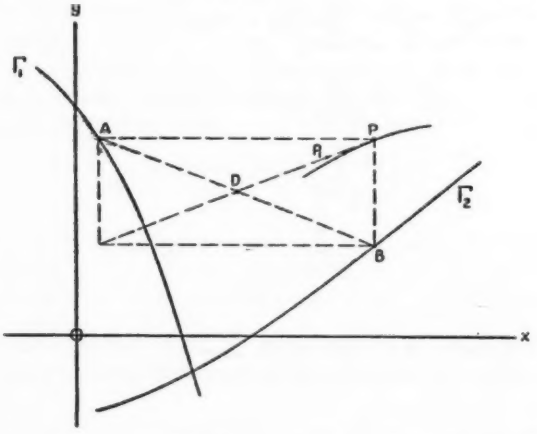
In a previous note (this MONTHLY, vol. 60, 1953, pp. 710–711), J. P. Russell extends the graphical method of Liénard [1] to solving differential equations of the form

$$\frac{dy}{dx} = \frac{\phi(y) + x}{\psi(x) + y}. \tag{1}$$

In this note, the method is extended with slight changes to differential equations of the form

$$\frac{dy}{dx} = \frac{\psi(x) + y}{\phi(y) + x}. \tag{2}$$

Here $\phi(y)$ and $\psi(x)$ are continuous functions.



One first plots the curves Γ_1: $[x = -\phi(y)]$, and Γ_2: $[y = -\psi(x)]$, as in the figure. Through an arbitrary point, $P(x, y)$, a line is drawn parallel to the x axis intersecting Γ_1 at A, and also, a line through P parallel to the y axis is drawn intersecting Γ_2 at B. One then draws the line PD, where D is the midpoint of segment AB. A small section of the integral curve of (2) passing through P is obtained by taking a small segment PP_1 of PD. The procedure is now repeated

starting from point P_1. The proximity of the successive points $P, P_1, P_2, \cdots$ taken will determine the accuracy of the construction.

The justification of the construction is as follows:

The coordinates of points A, B, and D are $[-\phi(y), y]$, $[x, -\psi(x)]$, and $[(x-\phi(y))/2, (y-\psi(x))/2]$, respectively. Thus the slope of PD is

$$\frac{\psi(x)+y}{\phi(y)+x}$$

as is required by equation (2).

Remark. If we take point D such that $AD/AB=K/(K+1)$, then the above method will give a graphical solution of the equation

$$\frac{dy}{dx}=K\frac{\psi(x)+y}{\phi(y)+x}. \tag{3}$$

For the special case $K=-1$, the line PD will be parallel to AB. But in this case, the equation can be solved by quadratures to

$$xy+\int\phi(y)dy+\int\psi(x)dx=0.$$

Reference

1. J. J. Stoker, Non-Linear Vibrations in Mechanical and Electrical Systems, Interscience, New York, 1950, pp. 31–36.

A NOTE ON INDETERMINATE FORMS

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It is not surprising that many students suspect the indeterminate form 0^0 to be equal to 1, believing that the elementary rules of algebra will apply. The example $x^{\alpha/\log x}$ immediately dispels this myth.

In attempting to construct examples of the indeterminate form 0^0, we immediately try $\lim_{x\to 0} x^{f(x)}$, where $f(0)=0$ and the derivative $f'(x)$ is assumed to exist in a neighborhood of the origin. But here one is led to an interesting limit and the surprising result that if $\lim_{x\to 0} x^{f(x)}$ exists it must be 1. Perhaps this word of caution will prevent lost time in looking for examples to remedy the students' suspicions.

We attempt to evaluate $\lim_{x\to 0} x^{f(x)}$ by following the text and take the logarithm, obtaining $\lim_{x\to 0}\dfrac{\log x}{1/f(x)}$. Here an application of l'Hospital's rule yields

for consideration

$$\lim_{x\to 0} \frac{f(x)}{x}\cdot\frac{f(x)}{f'(x)}.$$

If, at this point, we assume $f'(0)\neq 0$ and that $f'(x)$ is continuous, the $\lim_{x\to 0} f(x)/f'(x)$ exists and has the value 0; proving, of course, that $\lim_{x\to 0} x^{f'(x)}=1$.

The interesting observation to be made is the following:

Let $f(x)$ satisfy $f(0)=0$ and possess a derivative in a neighborhood of the origin. Then if $\lim_{x\to 0} f(x)/f'(x)$ exists it must have the value 0.

The italicized statement of course justifies the students' suspicions that $\lim_{x\to 0} x^{f'(x)}=1$, under the hypothesis on $f(x)$. The proof proceeds as follows:

Assume $\lim_{x\to 0} f(x)/f'(x)$ exists. Then 0 cannot be a limit point of zeros of $f'(x)$ or we are confronted with 0 being the limit point of points where $f(x)/f'(x)$ is either infinite or undefined. Hence we conclude that $f'(x)\neq 0$ for $0<|x|<\delta$. We immediately conclude that $f(x)\neq 0$ in the interval $0<x<\delta$, since otherwise $f'(\xi)=0$, for some ξ, $0<\xi<\delta$. Thus, without loss of generality, let $f(x)>0$ in the interval $0<x<\delta$.

If we now assume $\lim_{x\to 0} f(x)/f'(x)=L>0$, there exists a neighborhood δ' such that

$$\left.\begin{aligned} \frac{L}{2} &< \frac{f(x)}{f'(x)} < \frac{3L}{2} \\ \frac{2}{L} &> \frac{f'(x)}{f(x)} > \frac{2}{3L} \end{aligned}\right\} \quad \text{for } 0<x<\delta'.$$

Applying the Mean Value Theorem to $f(x)$ we find

$$\frac{f(x)}{f'(x)} = x\frac{f'(\xi)}{f'(x)} = x\cdot\frac{f(\xi)}{f(x)}\cdot\frac{f(x)}{f'(x)}\cdot\frac{f'(\xi)}{f(\xi)},$$

for x on the interval $0<x<\delta''$ ($\delta''\equiv$smaller of δ, δ') and for some ξ, $0<\xi<x$. We now observe that $f(x)/f'(x)$ is bounded by $3L/2$, $f'(\xi)/f(\xi)$ by $2/L$, and $f(\xi)/f(x)$ by 1 since $f'(x)$ must be >0. Therefore

$$\frac{f(x)}{f'(x)} < x\cdot 3$$

on $0<x<\delta''$; and consequently $\lim_{x\to 0} f(x)/f'(x)=0$, a contradiction to the assumption that $L\neq 0$. Hence $L=\lim_{x\to 0} f(x)/f'(x)$ must equal 0, and the student should be pleased since $\lim_{x\to 0} x^{f'(x)}=1$ as desired.

INVARIANTS AND THE GENERAL EQUATION OF THE SECOND DEGREE

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In the theory of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

the discussion of the invariance of $\Theta = A + C$, $\Phi = B^2 - 4AC$ and

$$\Delta = \frac{1}{2}\begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix},$$

under a rigid motion of the axes presents some difficulties for the elementary student. It is the purpose of this note to show that for classes familiar with the rudiments of the calculus a simple proof of invariance under a rotation of the axes is possible.

A rotation through the angle θ transforms the given equation into

$$a\bar{x}^2 + b\bar{x}\bar{y} + c\bar{y}^2 + d\bar{x} + e\bar{y} + f = 0$$

where

$$a = A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta = \tfrac{1}{2}[(A + C) + (A - C)\cos 2\theta + B\sin 2\theta],$$

$$b = 2(C - A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) = (C - A)\sin 2\theta + B\cos 2\theta,$$

$$c = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta = \tfrac{1}{2}[(A + C) + (C - A)\cos 2\theta - B\sin 2\theta],$$

$$d = D\cos\theta + E\sin\theta,$$

$$e = -D\sin\theta + E\cos\theta,$$

$$f = F.$$

We use primes to indicate differentiation with respect to θ:

$$a' = (C - A)\sin 2\theta + B\cos 2\theta = b$$

$$b' = 2(C - A)\cos 2\theta - 2B\sin 2\theta = 2(c - a)$$

$$c' = (A - C)\sin 2\theta - B\cos 2\theta = -b$$

$$d' = -D\sin\theta + E\cos\theta = e$$

$$e' = -D\cos\theta - E\sin\theta = -d$$

$$f' = 0.$$

Hence $\Theta' = a' + c' = 0$, and $\Phi' = 2bb' - 4ac' - 4a'c = 4[b(c - a) + ab - bc] = 0$ and so Θ and Φ are independent of θ, that is, invariant. Also, expanding the determinant in the formula for Δ by elements of the last row,

$$\Delta = \tfrac{1}{2}[d(be - 2cd) + e(bd - 2ae) - 2f\Phi] = bde - cd^2 - ae^2 - f\Phi.$$

Consequently,

$$\Delta' = b'de + bd'e + bde' - 2cdd' - c'd^2 - 2aee' - a'e^2$$
$$= 2(c - a)de + be^2 - bd^2 - 2cde + bd^2 + 2ade - be^2 = 0,$$

and so Δ is also invariant with respect to a rotation.

A similar proof of invariance with respect to a translation, $x = x' + x_0$, $y = y' + y_0$, is readily constructed, using partial derivatives with respect to x_0 and y_0. Here, however, the gain in simplicity is not so marked.

THE USE OF INDUCTION IN EXISTENCE PROOFS

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The use of induction in establishing the existence of sequences is illustrated by the proof of the following very well-known theorem:

Theorem. *Let $p_1, p_2, \cdots$ be a sequence of positive numbers, such that every open interval that includes 0 contains infinitely many of the numbers p_i. Then there is a descending sequence $q_1, q_2, \cdots$, such that*

(1) *$q_1, q_2, \cdots$ converges to 0, and*

(2) *every q_i is a term of the sequence $p_1, p_2, \cdots$.*

Lemma 1. *For each n, there is exactly one finite sequence Q_n: $q_1, q_2, \cdots, q_n$, satisfying the following conditions:*

(1) *the first term q_1 of Q_n is p_1.*

(2) *if $i < n$, then q_{i+1} is the first term of the sequence $p_1, p_2, \cdots$ which is less than $1/(i+1)$ and less than q_i.*

Proof of lemma. Let P_n be the proposition that there is exactly one sequence Q_n, satisfying (1) and (2) of the lemma. Then P_1 is obviously true. Also, P_n implies P_{n+1}, because given the unique Q_n: $q_1, q_2, \cdots, q_n$, there is exactly one way to define q_{n+1} so that the augmented sequence Q_{n+1}: $q_1, q_2, \cdots, q_n, q_{n+1}$ will satisfy (2) of the lemma. The lemma therefore follows by induction on n.

Lemma 2. *Let i be a positive integer. Let m and n be integers $\geqq i$. Let Q_m and Q_n be as in Lemma 1. Then the i-th term q_i of Q_m is the same as the i-th term q_i' of Q_n.*

Proof of lemma. Let Q_i be the sequence consisting of the first i terms, $q_1, q_2, \cdots, q_i$ of Q_m. Let Q_i' be the sequence consisting of the first i terms $q_1', q_2', \cdots, q_i'$ of Q_n. By Lemma 1, Q_i is the same as Q_i'. Therefore, in particular, $q_i = q_i'$, which was to be proved.

With the aid of these *unique finite sequences*, satisfying (1) and (2) of Lemma 1, we can now define an infinite sequence, of the sort required in the conclusion of the theorem. For each i, let the i-th term q_i of our infinite sequence be the positive number which is the i-th term q_i of *every* sequence Q_m for which $m \geqq i$.

We know that $q_1, q_2, \cdots$ converges to 0, because (by (2) of Lemma 1) $0<q_i<1/i$ for $i>1$.

In the above discussion, the treatment of sequences is informal. More precisely, a finite sequence Q_n (of n terms) is a function whose domain of definition is the set of all positive integers $\leqq n$; and an infinite sequence Q is a function whose domain of definition is the set of all integers $\geqq 1$. If a function is in turn defined in the usual way as a collection of ordered pairs, then the argument goes as follows: Let $Q=\cup Q_n$. By Lemma 1, every integer n occurs as the first term of at least one pair in Q. By Lemma 2, each such n occurs as the first term of at most one pair in Q. Therefore Q is a function defined over the positive integers, which was to be proved.

Note that the idea behind this proof can be stated roughly as follows: If we want to define a sequence, then we should explain what the first term of the sequence is to be, and we should give a rule whereby given the i-th term of the sequence, the $(i+1)$-st term is uniquely determined. This idea is used, without further analysis, even by writers of formidable rigor. The principal merit of the complete argument is not rigoristic but pedagogic. The basic device used in passing from the finite sequences to the infinite sequence is, in essence, the basic idea in the proof of the well-ordering theorem. (See Zermelo, *Beweis, dass jede Menge wohlgeordnet werden kann*, Math. Ann. vol. 59, 1904, pp. 514–516, and *Neuer Beweis für die Möglichkeit der Wohlordnung*, Math. Ann. vol. 65, 1907–08, pp. 107–128; or R. L. Moore, *Foundations of Point Set Theory*, Colloquium Publications of the Amer. Math. Soc. vol. 13, New York, 1932, pp. 1–4). The analogy becomes clearer if we abstract the principle used in the above proof, as follows:

THEOREM. *Let S be a collection of finite sequences, such that* (1) *for each positive integer n, S contains at least one sequence Q_n, having exactly n terms, and* (2) *if sequences Q_m, Q_n belong to S, and $m<n$, then Q_m is the same as the sequence consisting of the first m terms of Q_n. Then there is exactly one infinite sequence Q: $q_1, q_2, \cdots$, such that if Q_n belongs to S, then Q_n is the same as the sequence consisting of the first n terms of Q.*

The proof is trivial. Note that the collection S is very closely analogous to the collection of well-ordered sequences used in the proof of the well-ordering theorem; and note that in both cases the existence of the complete sequence is a consequence of the uniqueness of the partial sequences.

Obviously, none of the above is new. On the contrary, all of it is old, in a much more general form. But the proof of the well-ordering theorem is hard. It is the author's feeling that this proof is hard largely because it comes to grips, in the transfinite case, with a difficulty which in the finite case is almost universally ignored. If so, then rigorous proofs of the existence of simple sequences would be useful in preparing the student for the proof of the well-ordering theorem.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1106. *Proposed by C. I. Lubin, University of Cincinnati*

Two non-parallel, non-coincident lines which cut the circle $|z| = r$ in the points a, b and c, d respectively, where a, b, c, d are complex numbers not necessarily all different, intersect in the point z given by

$$z = (a^{-1} + b^{-1} - c^{-1} - d^{-1})/(a^{-1}b^{-1} - c^{-1}d^{-1}).$$

E 1107. *Proposed by Victor Thébault, Tennie, Sarthe, France*

On the edges AB, AC, AD of a tetrahedron $ABCD$ are marked points M, N, P such that $AB = nAM$, $AC = (n+1)AN$, $AD = (n+2)AP$. Show that the plane MNP contains a fixed line as n varies.

E 1108. *Proposed by René Bloch, Humanistisches Gymnasium, Basle, Switzerland*

Let n, k, $a+1$ be three positive integers which are not all odd. Express $\sum_{i=0}^{k}(n+ai)^3$ as a difference of two integer squares.

E 1109. *Proposed by Erich Michalup, Caracas, Venezuela*

Find numbers so that their squares, when reversed, are the squares of the reversed numbers.

E 1110. *Proposed by Edgar Karst, Independence, Missouri*

An *ideal* cryptarithm has only one solution and involves all ten digits. Solve the following ideal addition cryptarithm based upon the name of the daily newspaper, Hannoversche Presse:

$$\begin{array}{r} H\ A\ N\ N\ O\ V \\ E\ R\ S\ C\ H\ E \\ \hline P\ R\ E\ S\ S\ E \end{array}.$$

SOLUTIONS

Making Change

E 1076 [1953, 479]. *Proposed by L. A. Ringenberg, Eastern Illinois State College*

John has a one dollar bill and wishes to change it into cold hard cash, any combination of coins, even perhaps a silver dollar. Jacob has a \$100 bill and wants it changed into smaller bills; he wants at least five ones; he doesn't want all ones and he doesn't want any twos. In how many different ways can a bank teller make change for John? for Jacob?

Solution by Julian Braun, Washington, D. C. Suppose the teller gives John H half dollars, Q quarters, and D dimes. Then the teller can choose to give John any number of nickels N, such that $0 \leqq N \leqq 20-10H-5Q-2D$ after which the number of pennies is automatically determined. Each possible value of N in this case gives rise to a different way John can receive change, a total of $21-10H-5Q-2D$ ways. The maximum value of H is 2, that of Q is $4-2H$, and that of D is $[10-5H-2.5Q]$, where $[x]$ denotes the greatest integer not exceeding x. Thus John can receive change in

$$\sum_{H=0}^{2} \sum_{Q=0}^{4-2H} \sum_{D=0}^{[10-5H-2.5Q]} (21 - 10H - 5Q - 2D) + 1 = 293$$

ways, where the 1 has been added to allow for the silver dollar.

Similar reasoning for Jacob gives

$$\sum_{f=0}^{1} \sum_{w=0}^{4-2f} \sum_{t=0}^{1-5f-2w} (20 - 10f - 4w - 2t) - 1 = 293$$

ways, the same number as before! Here f is the number of fifties, w the number of twenties, and t the number of tens.

Also solved by Leon Bankoff, W. E. Buker, J. E. Darraugh, J. W. Hamblen, A. R. Hyde, Sam Kravitz, D. C. B. Marsh, Azriel Rosenfeld, C. W. Trigg, R. Z. Vause, and the proposer.

Scales of Notation

E 1077 [1953, 479]. *Proposed by J. A. Ward, University of Kentucky*

Let b be an integer greater than 2 and let $n = b^{b-1}-1$. Show that

(1) $n/(b-1)$ when converted to base b is a $(b-1)$-digit integer whose digits are all ones.

(2) $n/(b-1)^2$ when converted to base b is a $(b-2)$-digit integer whose digits from left to right are $1, 2, 3, \cdots, b-4, b-3, b-1$.

Solution by C. W. Trigg, Los Angeles City College. (1) We have

$$N = (b^{b-1} - 1)/(b - 1) = b^{b-2} + b^{b-3} + \cdots + b + 1.$$

To base b, the digits of N are the coefficients of the b^i, so they consist of $b-1$ ones.

(2) To find $M=N/(b-1)$ the division may be accomplished synthetically:

$$\begin{array}{cccccccc|c} 1 & 1 & 1 & 1\cdots & 1 & 1 & 1 & 1 & 1 \\ & 1 & 2 & 3\cdots & b-5 & b-4 & b-3 & b-2 & \\ \hline 1 & 2 & 3 & 4\cdots & b-4 & b-3 & b-2 & b-1 & \end{array}$$

Since the remainder of the division is $b-1$, the coefficient of the b^0 term in M is $b-2+(b-1)/(b-1)$ or $b-1$. With this modification the third line of the synthetic division gives the $b-2$ digits of M in order.

Also solved by Bonnie Baker, L. F. Boron, D. R. Clutterham, Fred Discepoli, J. R. Enterline, A. L. Epstein, B. A. Hausmann, R. W. Heath, Walter Hoffman, Vern Hoggatt, Douglas Holdridge, A. R. Hyde, John Jones, Jr., Ray Jurgensen, M. S. Klamkin, D. C. B. Marsh, R. J. Mercer, Ancel Mewborn, Marian Moore, Morris Morduchow, E. G. Musch, G. B. Parrish, M. J. Pascual, L. A. Ringenberg, Azriel Rosenfeld, Nathan Schwid, R. E. Shafer, D. R. Sudborough, T. H. Sumner, J. A. Tierney, Donald Trumpler, R. Z. Vause, Chih-yi Wang, Maud Willey, J. W. Young, and the proposer. Late solutions by Sybil Kirk, A. E. Livingston, T. M. Morrow, Margaret Olmsted, S. Parameswaran, S. H. Shugart, and A. V. Sylwester.

Maximizing a Certain Product

E 1078 [1953, 479]. *Proposed by N. Shklov, University of Saskatchewan*

Let a real positive number n be split into x equal parts in such a manner that the product of the parts will be greatest. How many parts will there be?

Solution by C. E. Miller, University of Saskatchewan. We are to find the maximum term of the sequence $\{T_m\}$, where $T_m=(n/m)^m$. It is easily established that the value of the continuous real variable x which makes the function $(n/x)^x$ a maximum is n/e. Hence if n/e is an integer it is the required value of m. Otherwise let $p=[n/e]$. By considering T_{p+1}/T_p we easily see that T_{p+1} is greater than, less than, or equal to T_p according as n is greater than, less than, or equal to $(p+1)^{p+1}/p^p$, and so the required value of m is $p+1$, p, or both, respectively.

Also solved by J. W. Baldwin, Leon Bankoff, L. F. Boron, Julian Braun, I. A. Dodes, Fred Discepoli, A. L. Epstein, Michael Goldberg, R. R. Gutzman, M. S. Klamkin, J. D. Haggard, J. W. Hamblen, J. R. Hatcher and S. L. Morris (jointly), Vern Hoggatt, A. R. Hyde, John Jones, Jr., Ray Jurgensen, J. G. Leghorn, D. C. B. Marsh, Beckham Martin, L. V. Mead, R. J. Mercer, Donald Miller, Morris Morduchow, C. S. Ogilvy, F. D. Parker, G. B. Parrish, M. J. Pascual, L. L. Pennisi and N. C. Scholomiti (jointly), L. A. Ringenberg, Nathan Schwid, R. E. Shafer, O. E. Stanaitis, J. A. Tierney, R. Z. Vause, G. W.

Walker, M. C. Wicht, Arthur Wormser, and the proposer. Late solutions by A. E. Livingston, Margaret Olmsted, and A. V. Sylwester.

Many of these solutions were incomplete.

Algebraic Treatment of Some Trigonometry Formulas

E 1079 [1953, 479]. *Proposed by V. L. Klee, Jr., University of Washington*

Suppose f and g are real functions such that always $g(x-y)=g(x)g(y)+f(x)f(y)$, and such that $f(t)=1$ and $g(t)=0$ for some $t\neq 0$. Prove that always $g(x\pm y)=g(x)g(y)\mp f(x)f(y)$ and $f(x\pm y)=f(x)g(y)\pm g(x)f(y)$.

Solution by T. S. Chihara, Purdue University. From the symmetric roles played by x and y in

$$g(x-y)=g(x)g(y)+f(x)f(y) \tag{1}$$

it follows that g is an even function. Setting $y=t$ in (1) gives

$$g(x-t)=f(x). \tag{2}$$

On the other hand if we set $y=x-t$ in (1) we obtain

$$g(x)g(x-t)+f(x)f(x-t)=0,$$

or, using (2),

$$f(x)[g(x)+f(x-t)]=0.$$

Since this last relation is true for all x we have

$$f(x-t)=-g(x). \tag{3}$$

Thus $f(x)=-g(x+t)=-g(-x-t)=-f(-x)$, so that f is an odd function. Thus the first desired relation follows immediately. To obtain the second relation we replace x by $x\pm y$ in (2) and expand, getting

$$f(x\pm y)=g[x\pm(y\mp t)]=g(x)g(y\mp t)\mp f(x)f(y\mp t).$$

Use of (2) and (3) now gives the desired result.

Also solved by J. W. Baldwin, Leonard Carlitz, P. L. Chessin, J. R. Cox, Edgar Dougherty, D. S. Greenstein, A. S. Gregory, J. R. Hatcher, Vern Hoggatt, Douglas Holdridge, R. W. Huff, C. E. Kerr, M. S. Klamkin, Sidney Kravitz, J. Lehner, E. J. McShane and the proposer (jointly), D. C. B. Marsh, F. D. Parker, G. B. Parrish, M. J. Pascual, L. L. Pennisi, J. D. Reid, B. E. Rhoades, L. A. Ringenberg, Azriel Rosenfeld, Nathan Schwid, O. E. Stanaitis, Donald Trumpler, R. M. Walter, L. E. Ward, Jr., and J. V. Whittaker. Late solutions by A. E. Livingston, Margaret Olmsted, S. Parameswaran, and W. D. Sørbyn.

Klamkin and Rosenfeld each pointed out that f and g are periodic of period $4t$. Reid and Gregory noted that the condition $t\neq 0$ is superfluous in the presence of the other hypotheses. Klamkin showed that if we further assume that f and

g are once differentiable, then $f(x)=\sin x$ and $g(x)=\cos x$. Pennisi established the same result under the assumptions that $f(x)$ and $g(x)$ are continuous, $\lim_{x\to 0} f(x)/x=1$, $\lim_{n\to\infty} f(x)x^n/n!=0$ for all values of x.

The problem shows that the usual formulas for $\cos(A\pm B)$ and $\sin(A\pm B)$ follow purely algebraically from the formula for $\cos(A-B)$ and the fact that $\sin \pi/2=1$, $\cos \pi/2=0$.

A Property of the Nine-Point Center

E 1080 [1953, 479]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let I be the incenter, N the nine-point center, and D the midpoint of side BC of a triangle ABC. Show that one of the common tangents to the circles $I(N)$ and $D(N)$ is parallel to BC.

Solution by Leon Bankoff, Los Angeles, Calif. One of the common tangents to the circles $I(N)$ and $D(N)$ is parallel to BC if the radius of the circle $D(N)$ is equal to the sum of the inradius of triangle ABC and the radius of circle $I(N)$. Since, by Feuerbach's Theorem, $IN=R/2-r$, where $R/2$ is the radius of the nine-point circle and r is the inradius of triangle ABC, the proposition is proved.

Also solved by W. B. Carver (using conjugate coordinates), L. V. Mead, D. M. Seward, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4578. *Proposed by N. S. Mendelsohn, University of Manitoba*

Find the number of essentially different eleven card Canasta hands which can be dealt from a 104 card pack. (We ignore red threes since they are always replaced in a hand. The pack consists of 8 aces, 8 twos, 4 threes, 8 fours, 8 fives, $\cdots$, 8 kings and 4 jokers. All cards of a given denomination are considered identical.)

4579. *Proposed by I. J. Schoenberg, University of Pennsylvania*

Let the relation

$$e^z = \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right)$$

hold for $|z| < r$, where $a_n \geqq 0$, $b_n \geqq 0$ $(n = 0, 1, \cdots)$. Show that the two factors on the right side must be entire functions of the form

$$e^{az+c}, \qquad e^{bz-c}, \qquad (a \geqq 0, b \geqq 0, a + b = 1).$$

4580. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

For what real values of a does the power series expansion of the function $(1-ax+ax^2-x^3)^{-1}$ have all its coefficients non-negative?

4581. *Proposed by Ky Fan, University of Notre Dame*

Let M be a metric space in which every closed spheroid, *i.e.*, set of the form

$$S(z_0; r) = \{z \in M \mid d(z_0, z) \leqq r\}, \qquad (z_0 \in M, r > 0)$$

is compact (d denotes the distance). Let f be a continuous transformation from M into itself. Suppose that there is a point $x_0 \in M$ such that

$$d(x_0, f(x)) < d(x_0, x) \tag{1}$$

for every $x \neq x_0$ of M. Prove

(i) If $f(x_0) = x_0$, then

$$\lim_{n\to\infty} f^n(x) = x_0 \tag{2}$$

for all $x \in M$; where $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$.

(ii) If $f(x_0) \neq x_0$, then for every point $x \in M$, there is a positive integer k, depending on x, such that $f^k(x) = x_0$.

4582. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Show that

$$\text{(a)} \qquad \sum_{n=1}^{p} \frac{S_{1,n}^2}{n} \sim \frac{5}{3}\sum_{n=1}^{\infty} \frac{1}{n^3} + \frac{1}{3}(\gamma + \log p)^3,$$

$$\text{(b)} \qquad \sum_{n=1}^{p} \frac{S_{2,n}}{n} \sim \frac{\pi^2}{6}(\gamma + \log p) - \sum_{n=1}^{\infty} \frac{1}{n^3},$$

where γ is Euler's constant and

$$S_{m,n} = \sum_{r=1}^{n} \frac{1}{r^m}.$$

SOLUTIONS

A Special Case of Egyptian Fractions

4512 [1952, 640]. *Proposed by E. P. Starke, Rutgers University*

Prove that every positive rational number with odd denominator is a sum of a finite number of distinct terms from the sequence $1/3, 1/5, 1/7, 1/9, \cdots$.

Solution by Robert Breusch, Amherst College.

LEMMA 1: *If* $A=3^n5$, $n\geqq 1$, *and* $B<3^n$, *then* B/A *is a sum of distinct positive fractions of the form* $1/3^t5$ *or* $1/3^t$ $(t\geqq 1)$.

Proof: Let $B=a_0+3a_1+\cdots+3^{n-1}a_{n-1}$, with $0\leqq a_r\leqq 2$. Then

$$\frac{B}{A}=\frac{a_0}{3^n5}+\frac{a_1}{3^{n-1}5}+\cdots+\frac{a_{n-1}}{3\cdot 5}.$$

If $a_{n-1}=2$, write $2/3\cdot 5=1/3^2+1/3^25$, and add the last term to $a_{n-2}/3^25$. Thus we have to consider now $b/3^25$, where $0\leqq b\leqq 3$. If $b=3$, this is $1/15$, a term which in this case has not occurred before. If $b=2$, proceed as before: $2/3^25=1/3^3+1/3^35$, and now combine the last term with $a_{n-3}/3^35$. A continuation of this procedure leads to the desired representation of B/A.

LEMMA 2. *If* $A=3^n5^m$, $n\geqq 2$, $m\geqq 1$, *and* $B<3^n5^{m-1}$, *then* B/A *is a sum of distinct positive fractions of the form* $1/3^r5^s$.

Proof: Let $B=a_0+5a_1+5^2a_2+\cdots+5^{m-1}a_{m-1}$, with $0\leqq a_t\leqq 4$ for $t\leqq m-2$, and $a_{m-1}<3^n$. Then

$$\frac{B}{A}=\frac{a_0}{3^n5^m}+\frac{a_1}{3^n5^{m-1}}+\cdots+\frac{a_{m-2}}{3^n5^2}+\frac{a_{m-1}}{3^n5}.$$

By Lemma 1, the last term is a sum of fractions of the desired kind, and with denominators that contain at most the first power of 5. If a_{m-2} is $4=3+1$, or 3, then $a_{m-2}/3^n5^2$ leads to one or two fractions whose numerator is 1, and whose denominator contains exactly the second power of 5. If $a_{m-2}=2$, then $2/3^n5^2=1/3^{n-2}5^3+1/3^n5^3$. Combining the last term with $a_{m-3}/3^n5^3$, we have to consider now $b/3^n5^3$, $(0\leqq b\leqq 5)$. For $b=5$, this is $1/3^n5^2$, a fraction which in this case has not occurred previously. For $b=4$ or $b=3$, we get one or two fractions whose denominators contain the third power of 5. If $b=2$, proceed as before. Continuing in this way, we get the desired representation of B/A.

LEMMA 3: *Let* $p_1=3, p_2, \cdots, p_n, \cdots$ *be the odd primes. Let* $A=p_1^{s_1}p_2^{s_2}\cdots p_n^{s_n}$, *with* $s_1\geqq 2$, *and* $s_t\geqq 1$ *for* $2\leqq t\leqq n$. *Let* B *be less than* A/p_n. *Then* B/A *is a sum of distinct fractions of numerator* 1, *whose denominators contain no prime factors greater than* p_n.

Proof: by induction on n. For $n=2$ the statement is true by Lemma 2. Assume it has been proved up to $n-1$. Let

$$B = a_0 + a_1 p_n + \cdots + a_{s_n} p_n^{s_n}, \quad \text{with} \quad 0 \leqq a_t < p_n \quad \text{for} \quad t \leqq s_n - 1$$

and $a_{s_n} < A/p^{s_n+1}$.
Thus, writing $A = Cp_n^{s_n}$, $a_{s_n} < C/p_n < C/p_{n-1}$, and

$$\frac{B}{A} = \frac{a_0}{C}\frac{1}{p_n^{s_n}} + \frac{a_1}{C}\frac{1}{p_n^{s_n-1}} + \cdots + \frac{a_{s_n}}{C}.$$

Now each numerator is certainly less than C/p_{n-1}, because $p_n < p_1 p_2 \cdots p_{n-1}$ for $n \geqq 4$; while for $n=3$, $p_n=7$ is less than $3^2 \leqq C/5$. Thus each of the partial fractions for B/A can be represented in the required form, and no two fractions will have any term in common, because their denominators contain different powers of p_n.

THEOREM: *Let A be odd; then B/A can be written as a sum of distinct positive fractions with odd denominators and unit numerators.*

Proof: Let p_n be the greatest prime factor of A. Write the fraction in such a form that its denominator satisfies the hypothesis of Lemma 3. If the resulting numerator is less than A/p_n, the proof is complete from Lemma 3. Otherwise let $m > n$ be such that

$$\sum_{t=n+1}^{m} \frac{1}{p_t} < \frac{B}{A} < \sum_{t=n+1}^{m+1} \frac{1}{p_t}.$$

Then $B'/A' = B/A - \sum_{t=n+1}^{m} 1/p_t$ satisfies all the conditions of Lemma 3; each term in the representation of B'/A' is less than $1/p_m$, and therefore B/A itself is now expressed in the desired form.

Also solved by E. P. Churchill, and B. M. Stewart.

Editorial Note. It is easy to see that B/A has infinitely many representations as a sum of odd unit fractions since it can be broken up into $B_1/A_1 + B_2/A_2 + \cdots$ in innumerable ways and then the above method applied to each part.

The method rarely provides the simplest sum of unit fractions. For example $2/13 = 1/7 + 1/91$ but the method gives the sum of the reciprocals of 15, 17, 45, 315, 495, 1155, 14015, and 153153. The interesting problem of a "best" representation is not considered here.

The other solvers made the proof depend upon the following theorem. *Let A be any odd number, then there exist infinitely many odd multiples M_i of A, each of which has the property: every integer n such that $2 < n < \sigma(M_i) - 2$, is a sum of distinct positive divisors of M_i.*

Polynomials Suggested by Familiar Integers

4513 [1952, 701]. *Proposed by Leonard Carlitz, Duke University*

Put $\{m\}!=(x^m-1)(x^{m-1}-1)\cdots(x-1)$, $\{0\}!=1$.

A. Show that

$$\frac{\{2m\}!\{2n\}!}{\{m+n\}!\{m\}!\{n\}!} \qquad (m, n \geqq 0) \tag{1}$$

is a polynomial with rational coefficients.

B. The quotient

$$\frac{\{mn\}!}{\{m\}!(\{n\}!)^m}(x-1)^m$$

is not integral for all $m, n \geqq 0$. How can it be modified to be integral and at the same time reduce to $(mn)!/m!(n!)^m$ for $x=1$?

Solution by J. V. Whittaker, University of California, Los Angeles.

A. Let $F_k(x)$ be the (irreducible) cyclotomic polynomial of index k. The highest power of $F_k(x)$ which divides $\{m\}!$ is obviously $[m/k]$, the integral part of m/k. The rational function in question, then, is a polynomial with integral coefficients if, and only if,

$$\left[\frac{2m}{k}\right]+\left[\frac{2n}{k}\right] \geqq \left[\frac{m+n}{k}\right]+\left[\frac{m}{k}\right]+\left[\frac{n}{k}\right],$$

for every k. Setting $\mu=m/k-[m/k]$ and $\nu=n/k-[n/k]$, the above inequality reduces to $[2\mu]+[2\nu] \geqq [\mu+\nu]+[\mu]+[\nu]$ which obviously holds.

B. The highest power of $F_k(x)$ which divides $\{mn\}!/\{m\}!(\{n\}!)^m$ is

$$\left[\frac{mn}{k}\right]-\left[\frac{m}{k}\right]-m\left[\frac{n}{k}\right].$$

Setting $m=ak+r$ and $n=bk+s$, $0 \leqq r, s<k$, the above expression reduces to $[rs/k]+as-a$ which is <0 if, and only if, $s=0$, and then it is $-a$. Therefore

$$P(x)=\frac{\{mn\}!(x-1)^m}{\{m\}!(\{n\}!)^m}\prod_{1<k\mid n}\left\{\frac{F_k(x)}{F_k(1)}\right\}^{[m/k]}$$

is a polynomial with rational coefficients such that $P(1)=(mn)!/m!(n!)^m$.

Also solved by O. E. Stanaitis and the Proposer.

Solutions of an Integral Equation

4514 [1952, 702]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

$\psi(t)=$constant and $\psi(t)=|t|$ are solutions of the integral equation

$$2\psi(t)=\int_{-1}^{1}\psi(2tx)dx,$$

as is easily verified. Are there any other solutions?

Solution by J. E. Wilkins, Jr., Nuclear Development Associates, Inc. White Plains, N. Y. There are others. To find additional solutions, substitute $|t|^a$ for $\psi(t)$. This function will satisfy the integral equation if and only if $2^a=a+1$. This transcendental equation has exactly two real roots, namely 0 and 1, and has infinitely many complex roots

$$\alpha_n \pm i\beta_n(\log 2)^{-1}$$

such that

$$\alpha_n = -1+\beta_n(\log 2)^{-1}\cot\beta_n,$$

$$2\beta_n(\log 2)^{-1}\exp(-\beta_n\cot\beta_n)=\sin\beta_n,$$

$$2n\pi<\beta_n<(4n+1)\pi/2 \qquad (n=1,2,\cdots).$$

For example, $\beta_1=7.454087$, $a_1=3.545368\pm 10.75397i$.

Also solved by G. A. Baker, Jr., R. H. Boyer, A. E. Currier and R. P. Bailey, L. L. Pennisi, Edgar Reich, H. J. Zimmerberg, and the Proposer.

Editorial Note. As shown also by the other solvers, there exist no real solutions of the integral equation, under the assumption that $\psi(x)$ is of class C^1 on $[0, 1]$, other than linear combinations of the solutions cited by the Proposer.

A Number Theory Summation

4515 [1952, 702]. *Proposed by Karl Goldberg, National Bureau of Standards, Washington, D. C.*

Prove

$$\sum x\equiv 2^{-4}+(2^{p-1}-1)/p \pmod p,$$

where p is an odd prime and the summation extends over all integers $x<p$ which are greater than $(p-1)/2$ and whose inverses mod p are also greater than $(p-1)/2$.

Solution by J. V. Whittaker, University of California, Los Angeles. The binomial expansion of $(1+1)^p$ gives

$$\frac{2^p-2}{p} = 1 + \frac{p-1}{2} + \frac{p-1}{2}\cdot\frac{p-2}{3} + \cdots + \frac{p-1}{2}\cdots\frac{p-(p-2)}{p-1}$$

$$\equiv 1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{p-1}$$

$$\equiv -2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{p-1}\right)$$

$$\equiv -\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{(p-1)/2}\right) \pmod{p}.$$

We have also

$$1 + 2 + 3 + \cdots + \frac{p-1}{2} = \frac{1}{2}\cdot\frac{p-1}{2}\cdot\frac{p+1}{2} \equiv -\frac{1}{8} \pmod{p}.$$

Hence

$$\frac{2^p-2}{p} + \frac{1}{8} \equiv -\sum\left(x + \frac{1}{x}\right) \pmod{p},$$

where the summation extends over the first half of the integers from 1 to $p-1$. Now if x_1 and $1/x_1$ both lie in the first half, then $x_2 \not\equiv -x_1$ and $1/x_2$ are both in the second half. Moreover, if x_1 is in the first half and $1/x_1$ in the second, then $x_2 \not\equiv 1/x_1$ is in the first half and $1/x_2$ in the second, so that the sum of all terms in $\sum(x+1/x)$ with x in the first half and $1/x$ in the second, is zero. Therefore

$$\frac{2^p-2}{p} + \frac{1}{8} \equiv -2\sum x \equiv 2\sum y \pmod{p},$$

where the first summation extends over integers in the first half with inverses in the first half, and the second summation over integers in the second half with inverses in the second half.

Also solved by Leonard Carlitz and the Proposer.

Editorial Note. Carlitz remarks that the same method leads to the formula

$$\sum x \equiv \frac{1}{8} + \frac{2^{p-1}-1}{2p} \pmod{p},$$

where the summation is restricted to such x that both x and its inverse are odd.

Logarithmic Concavity

4517 [1952, 702]. *Proposed by G. G. Lorentz, Wayne University*

A sequence a_n, $n=0, 1, \cdots$, is called logarithmically concave if

$$a_n^2 - a_{n-1}a_{n+1} \geqq 0, \qquad n = 1, 2, \cdots.$$

Prove that if the sequences a_n, b_n are positive and logarithmically concave, then their convolution $c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0$ has the same properties.

Solution by the Proposer. If a_n is logarithmically concave and positive then a_{n+1}/a_n is decreasing, hence for any such sequence

$$A_{pq} = a_pa_q - a_{p-1}a_{q+1} \geqq 0, \qquad p \leqq q \tag{1}$$

for $p, q \geqq 0$. If we put $a_n = 0$ for $n < 0$, (1) becomes correct for all $p \leqq q$. Therefore our result is implied by the following: *If the sequences* a_n, b_n, $n = \cdots, -1, 0, 1, \cdots$ *have the property* (1) *and if all series* $c_n = \sum_{\gamma=-\infty}^{+\infty} a_{n-\gamma}b_\gamma$, $n = \cdots, -1, 0, 1, \cdots$ *are absolutely convergent, then also the* c_n *have the property* (1).

From (1) it follows that $A_{pq} \leqq 0$ for $p > q$. We introduce also the B_{pq} for the sequence b_n. Then

$$\begin{aligned} c_n^2 - c_{n-1}c_{n+1} &= \Big(\sum_\nu a_{n-\nu}b_\nu\Big)^2 - \sum_\nu a_{n-1-\nu}b_\nu \sum_\mu a_{n+1-\mu}b_\mu \\ &= \sum_{\nu,\mu=-\infty}^{+\infty} (a_{n-\nu}a_{n-\mu} - a_{n-\nu-1}a_{n-\mu+1})b_\nu b_\mu \\ &= \sum_{\nu,\mu} u_{\nu\mu} = \frac{1}{2}\sum_{\nu,\mu}(u_{\nu\mu} + u_{\mu-1,\nu+1}) \geqq 0, \end{aligned}$$

since

$$\begin{aligned} u_{\nu\mu} + u_{\mu-1,\nu+1} &= (a_{n-\nu}a_{n-\mu} - a_{n-\nu-1}a_{n-\mu+1})(b_\nu b_\mu - b_{\nu+1}b_{\mu-1}) \\ &= A_{n-\nu,n-\mu}B_{\mu\nu} \geqq 0, \end{aligned}$$

which proves our statement.

The proposed result ceases to be correct if it is assumed only that $a_n \geqq 0$, $b_n \geqq 0$. For instance, if $a_0 = a_5 = b_2 = b_5 = 1$, and $a_n = b_m = 0$ for the remaining n, m, then $c_5 = c_7 = 1$, $c_6 = 0$, and c_n is not logarithmically concave. The statement proved in the solution remains true if we replace (1) by (1′): $A_{pq} \leqq 0$ for $p \leqq q$ and concavity by convexity. On the contrary, the result proposed is not true after this change, even if a_n, b_n are positive. For instance, $a_n = 1$, $b_n = q^n$, $q > 0$, then $c_n = (q^{n+1} - 1)/(q - 1)$ is not logarithmically linear and is logarithmically concave by the result, hence c_n is not logarithmically convex.

See also Kaluza, *Math. Zeitschrift* vol. 28, 1928, pp. 161–170, and Davenport and Pólya, *Canad. J. Math.*, vol. 1, 1949, pp. 1–5.

Also solved by Leonard Carlitz and O. E. Stanaitis.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Theory of Functions of a Complex Variable. By W. J. Thron, New York, John Wiley and Sons, 1953. 9+230 pages. $6.50.

This is a carefully written book which fills a need for a rigorous treatment of functions of a complex variable from a modern point of view. The purpose of the author to write a self-contained volume "in which all results are derived from a simple set of axioms" is admirably achieved. As a result, the book should be very useful as a text for graduate majors in mathematics.

The book is not designed for those people whose interest in complex variables is one of formal manipulation. For this reason it is probably not suitable as an introduction to the subject for students who wish simply to add another "mathematical tool" to their collection. Indeed, the contents will probably be more meaningful to any student who has at least a smattering of the theory already available. Although it is true, as stated, that no previous mathematical knowledge is required of the reader, it is also true that he should have a reasonable amount of mathematical maturity.

This in no sense limits the usefulness of the book for the purpose for which it is intended. Its usefulness may be even broader. Because of the range of topics covered, the book might well serve as a text in courses other than those treating complex variables. The first fourteen chapter headings suggest the possible flexibility in this regard: 1. Fundamental Concepts, 2. Real Numbers, 3. Cardinal Numbers, 4. Complex Numbers, 5. Sums and Products, 6. Hausdorff spaces, 7. Metric Spaces, 8. The Plane of Complex Numbers, 9. Limits, Continuity, Differentiability, 10. Real Functions of Real Variables, 11. Curves and Regions in the Plane of Complex Numbers, 12. Some Combinatorial Topology, 13. Jordan Curves, 14. Rectifiable and Directed Curves.

These constitute less than half the book, the remaining chapters being devoted to subjects more strictly related to complex analysis.

This reviewer would like to see more illustrative examples of such things as Hausdorff spaces which might help initiate the newcomer to the more abstract concepts. It is perhaps asking too much in a book which treats so many subjects, but the object might have been achieved by the inclusion of a wider variety of exercises in addition to those which simply ask for proofs of theorems.

All in all, however, this book should be a welcome addition to the literature both as a text and as a convenient reference.

W. D. MUNRO
University of Minnesota

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SUMMER CONFERENCE FOR HIGH SCHOOL TEACHERS

Under the sponsorship of the National Science Foundation, the University of Washington will hold a Summer Conference for High School Teachers of Mathematics in Seattle from July 26 to August 20. Principal speakers will be Professor B. W. Jones of the University of Colorado and Professor C. B. Allendoerfer of the University of Washington. A number of short lecture series, discussion groups, and field trips will complete the program. A limited number of financial grants will be available to help participants cover the cost of travel and subsistence. Interested teachers should write to: Office of Short Courses and Conferences, University of Washington, Seattle 5, Wash.

SIXTH ANNUAL INSTITUTE FOR TEACHERS AND PROFESSORS OF MATHEMATICS

The sixth institute sponsored by the Association of Teachers of Mathematics in New England will be held at the Massachusetts Institute of Technology from August 19–26, 1954. Group leaders include Professor Elmer Mode, Boston University; Mr. Edmund Berkeley, Berkeley Associates; Mr. Charles Mergendahl, Newton High School; Mr. Carl Shuster, Trenton State Teachers' College, and Professor Herman Shea, M. I. T. For full details write to Mrs. M. Isabelle Savides, Levi F. Warren Junior High School, West Newton, Mass.

PERSONAL ITEMS

University of Ottawa announces the following appointments: Professor Olivier Biberstein, Professor J. T. Duprat, Assistant Professor Leopold Vachon, Lecturer J. L. Allard, and Lecturer Yvon Grandchamp.

Mr. Walter Abramowitz, formerly research fellow at Polytechnic Institute of Brooklyn, is employed as a physicist at the Solid State Research Institute, New York City.

Mr. A. C. Andersen, who has been teaching at Ben Franklin High School, Rochester, New York, has accepted a position as an instructor at General Motors Institute, Flint, Michigan.

Mr. W. P. Anderson, formerly a graduate student at the University of Minnesota, has been appointed to an instructorship in the General College of the University.

Mr. F. J. Arena of North Dakota Agricultural College has been promoted to an assistant professorship.

Mr. E. H. Auerbach, previously a student at Columbia University, has been appointed Lecturer at the University.

Assistant Professor G. B. Ax of Virginia Military Institute has been promoted to an associate professorship.

Mr. Souren Babikian of Aleppo College, Syria, is teaching at Macalester College on an exchange visitor program.

Mr. Howard Baeumler of the University of Buffalo has been appointed to an assistant professorship at Marshall College.

Mr. J. J. Bailey is now Chairman of the Department of Mathematics of Shattuck School, Faribault, Minnesota.

Mr. H. W. Becker has a position as Transmitter Engineer with Station WOW, Omaha, Nebraska.

Mr. T. H. Bedwell, formerly production manager and engineer with the Freeman Company, Yankton, South Dakota, has been appointed to an instructorship at Southern State Teachers College, Springfield, South Dakota.

Mr. D. C. Benson has been appointed to a research assistantship at Stanford University.

Dr. R. L. Blair has been appointed to an instructorship at the University of California at Davis.

Assistant Professor H. D. Block of Iowa State College has been appointed Visiting Assistant Professor in the Department of Mathematics, Institute of Technology, University of Minnesota.

Mr. A. P. Boblétt has been promoted to the position of Chief of the Statistics Division, United States Naval Ammunition Depot, Crane, Indiana.

Mr. J. E. Bosshart, formerly a lecturer at Western Reserve University, has been appointed to an assistant professorship at the University of Dayton.

Dr. Leila D. Bram, previously a research associate at the University of Chicago, has accepted a position as a mathematician at the Office of Naval Research, Washington, D. C.

Mr. G. U. Brauer has been appointed to an instructorship in the Department of Mathematics of the Institute of Technology, University of Minnesota.

Mr. R. B. Bredemeier, recently a graduate assistant at Oregon State College, is teaching at Hermiston High School, Oregon.

Mr. R. C. Brown, Jr., formerly a part-time instructor at the University of Kentucky, has been appointed to an instructorship at Glenville State College.

Dr. R. J. Buehler has returned to his position as a staff member of the Sandia Corporation, Albuquerque, New Mexico.

Dr. L. P. Burton of the University of California at Davis has been promoted to an assistant professorship.

Mrs. Jewell H. Bushey of Hunter College has been promoted to a professorship.

Dr. G. C. Byers has been appointed to an assistant professorship at Michigan College of Mining and Technology.

Dr. Buchanan Cargal of Iowa State College has accepted a position as Senior Aerophysics Engineer with Consolidated Vultee Aircraft Corporation, Fort Worth, Texas.

Mr. A. J. Carlan has been appointed Research Physicist by the American Optical Company, Southbridge, Massachusetts.

Mrs. Audrey M. Carlan has a position as Mathematician with the American Optical Company, Southbridge, Massachusetts.

Mr. T. F. Carroll, formerly an assistant engineer with Celotex Corporation, Marrero, Louisiana, is now Quality Control Supervisor for Sefton Fibre Can Company, New Orleans, Louisiana.

Associate Professor C. E. Clark of Emory University has accepted a position as Operations Analyst in the Operations Research Office, Johns Hopkins University.

Dr. K. G. Clemans of the University of Oregon has a position as Statistician at the Naval Ordnance Test Station, China Lake, California.

Mr. K. J. Cohen has been appointed Carl Lotus Becker Fellow at Cornell University.

Mr. H. B. Coleman has accepted a position as Senior Engineer in the Research Laboratory of Bendix Aircraft Corporation, Detroit, Michigan.

Miss Margaret J. Cotter, previously a student at State Teachers College, Upper Montclair, New Jersey, is teaching at West Orange High School, New Jersey.

Mr. J. W. Creely, formerly a research chemist with American Cyanamid Company, Bound Brook, New Jersey, has accepted a position as Assistant Technical Director of the Heat and Mass Flow Analyzer Laboratory, Columbia University.

Mr. P. M. Curran of Fordham University has been promoted to an assistant professorship.

Dr. Elizabeth H. Cuthill, formerly a part-time instructor at the University of Maryland, has received an appointment as a mathematician at the David Taylor Model Basin, Carderock, Maryland.

Mr. T. S. Dean is now with Dean and Jordan, Architects, Dallas, Texas.

Mr. E. G. Douglas of the University of South Carolina has been appointed Professor and Head of the Department of Mathematics of Newberry College.

Assistant Professor J. E. Eaton of Queens College has been promoted to an associate professorship.

Mr. Karl Eide, previously at Emmanuel Missionary College, Berrien Springs, Michigan, has been appointed to a professorship at Madison College, Tennessee.

Dr. Paul Erdös has been appointed Visiting Professor at the University of Notre Dame.

Assistant Professor A. B. Farnell of the United States Military Academy has accepted a position as Research Engineer with North American Aviation, Los Angeles, California.

Miss Catherine S. Feeley, who has been a graduate student at the University of Notre Dame, is now an auditor with Northern State Company, Chicago, Illinois.

Dr. I. K. Feinstein of the University of Illinois, Navy Pier, Chicago, has been promoted to an assistant professorship.

Associate Professor F. N. Fisch is on leave of absence from Colorado State College of Education for the academic year 1953–54 and is at George Peabody College for Teachers.

Dr. W. T. Fishback of the University of Vermont has been appointed to an assistant professorship at Ohio University.

Dr. Harley Flanders of the University of California has been promoted to an assistant professorship.

Mr. G. C. Francis has been appointed Lecturer at Columbia University.

Miss Frances Freese of Mount Union College has been appointed Assistant Dean of Women and Assistant Professor of Psychology at Texas Technological College.

Mr. A. L. Gilmore, Jr., formerly an instructor at Pearl River Junior College, Poplarville, Mississippi, is now Radar Instructor at Keesler Air Force Base, Biloxi, Mississippi.

Reverend F. J. Ginivan has been appointed to an associate professorship at St. Francis Xavier University, Nova Scotia.

Mr. M. A. Glatt of the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, has accepted a position as a mathematician with General Electric Advanced Electronics Center, Ithaca, New York.

Mr. B. T. Goldbeck, Jr., of Texas Christian University has been appointed to an instructorship at the University of Oklahoma.

Miss Bernice Goldberg, previously a mathematician at the Air Force Research Center, Cambridge, Massachusetts, is now Technical Engineer in Numerical Analysis with the General Electric Company, Evandale, Ohio.

Mr. M. L. Goldwater, formerly an electronics engineer with Hughes Aircraft Company, Culver City, California, has accepted a position as Engineer with Librascope, Inc., Glendale, California.

Mr. K. E. Gorsline, who has been teaching at East High School, Denver, Colorado, has been appointed Assistant Principal of Merrill Junior High School, Denver, Colorado.

Dr. Arthur Grad, previously at the Office of Naval Research, Washington, D. C., has been appointed Mathematician at the Institute of Mathematical Sciences, New York University.

Mr. D. S. Greenstein, who has been a graduate student at the University of Pennsylvania, has been appointed to an assistant instructorship in the Moore School of Electrical Engineering of the University.

Mrs. Florence N. Greville, formerly an analyst in the Department of Defense, Washington, D. C., is teaching at Escala Americana de Rio de Janeiro.

Miss Louisa S. Grinstein, recently a graduate student at the University of Buffalo, has been appointed to a teaching fellowship at the University of Michigan.

Assistant Professor B. H. Gundlach of the University of Arkansas has been promoted to an associate professorship.

Mr. J. W. Hamblen has been appointed to a research assistantship in the Statistical Laboratory of Purdue University.

Mr. E. E. Hammond, Jr., has been appointed to an instructorship at Phillips Academy, Andover, Massachusetts.

Associate Professor Louise F. Hanson of Olivet College has been promoted to a professorship.

Assistant Professor Heinz Helfenstein of Stanford University has been appointed Lecturer at the University of Alberta.

Dr. I. N. Herstein, who has been with the Cowles Commission, University of Chicago, has been appointed to an assistant professorship at the University of Pennsylvania.

Mr. V. E. Hoggatt, Jr., has been appointed to an instructorship at San Jose State College.

Assistant Professor F. L. Holzhauser of Kent State University is teaching now at Shaker Heights High School, Cleveland, Ohio.

Dr. Burrowes Hunt has been appointed to an assistant professorship at Reed College.

Mrs. Verba W. Iturralde, who has been teaching at Bowie High School, El Paso, Texas, has been given an appointment at El Paso Technical Institute.

Mr. R. D. Johnson, Jr., has been appointed to a part-time instructorship at the University of Virginia.

Mr. W. W. Johnson of Huntington Polytechnic Institute has been appointed to an instructorship at Fenn College.

Mr. Raymond Kassler, previously mathematician at Evans Signal Laboratory, Belmar, New Jersey, has accepted a position as Development Engineer with R.C.A. Victor Division, Camden, New Jersey.

Miss Dora E. Kearney has been appointed to an associate professorship at Upper Iowa University, Fayette, Iowa.

Associate Professor E. S. Kennedy of American University of Beirut, Lebanon, was Visiting Professor at Brown University for the first semester of 1953–54 and is a member of the Institute for Advanced Study for the second semester.

Mr. R. W. Klopfenstein of Iowa State College has accepted a position as Research Engineer at R.C.A. Laboratories, Princeton, New Jersey.

Miss Mary B. Lieberknecht of Iowa State College has a position with the Boeing Airplane Company, Seattle, Washington.

Mr. F. H. Lloyd has been appointed to an instructorship at Missouri School of Mines and Metallurgy.

Mr. L. S. Lockingen of the University of Houston has been awarded a National Science Foundation Fellowship and is at the University of Texas.

Mr. H. C. Mayer, Jr., has been appointed Assistant Supervisor of Education, General Electric Company, Richland, Washington.

Assistant Professor Janet McDonald of Vassar College has been promoted to an associate professorship.

Mr. K. A. McGown, previously chairman of the Department of Mathematics of Wallington High School, New Jersey, has been appointed to an instructorship at Lafayette College.

Mr. R. L. Mentzer of North Dakota Agricultural College has been appointed to an instructorship at Teachers College of Connecticut.

Mr. J. W. Mettler, recently with the Educational Testing Service, Princeton, New Jersey, has accepted a position with the De Laval Steam Turbine Company, Trenton, New Jersey.

Mr. V. A. Miculka, formerly a graduate assistant at the University of Oklahoma, has been appointed Chairman of the Department of Mathematics of Frank Phillips College.

Mr. Akeley Miller has been appointed to an instructorship in the Physics Department of the University of South Dakota.

Associate Professor W. I. Miller of Bucknell University has been promoted to a professorship.

Assistant Professor Emeritus W. M. Miller of Washington and Lee University has been appointed to an associate professorship at Roanoke College.

Mr. L. J. Montzingo, Jr., of Roberts Wesleyan College has been promoted to an assistant professorship.

Professor G. E. Moore of the University of Illinois has been appointed Associate Dean of the College of Liberal Arts.

Miss Marian A. Moore, recently an instructor at Purdue University, is now Chairman of the Department of Mathematics, Glenbrook High School, Northbrook, Illinois.

Dr. K. B. Morgan has been appointed Chairman of the Department of Mathematics of Mount Kisco High School, New York.

Mr. J. T. Morse has been awarded a National Science Foundation Fellowship and is at Massachusetts Institute of Technology.

Mr. P. M. Moskowitz, previously a research engineer with Hiller Helicopters, Palo Alto, California, is now a dynamics engineer with Sikorsky Aircraft, Bridgeport, Connecticut.

Mr. G. R. Mott, formerly a student at Hofstra College, has accepted a position as Research Engineer with Republic Aviation Company, Farmingdale, New York.

Assistant Professor B. H. Mount, Jr., of the University of Pittsburgh has a position as Engineer with Westinghouse Electric Corporation, East Pittsburgh, Pennsylvania.

Assistant Professor J. H. Mulligan of New York University has been promoted to the position of Professor and Chairman of the Department of Mathematics.

Mr. H. E. Murray of American College, Tarsus, Turkey, has been promoted to the position of Chairman of the Department of Mathematics.

Mr. A. B. Neale, previously with Marquardt Aircraft Company, Van Nuys, California, is employed now as Mechanical Engineer by Hydro-Aire, Inc., Burbank, California.

Mr. J. A. Nickel has been appointed to an instructorship at Willamette University.

Mr. M. L. Norden, formerly mathematician in the Operations Research Office, Johns Hopkins University, has been appointed Senior Associate of Dunlap and Associates, Inc., Stamford, Connecticut.

Mr. L. R. Norwood, recently at the United States Army Signal Corps Laboratory, Fort Monmouth, New Jersey, is now Senior Engineer at the Electronic Defense Laboratory, Mountain View, California.

Mr. H. A. Osborn has been appointed Research Associate at the University of Chicago.

Mr. A. D. Pierson of Kansas City Junior College has retired.

Mr. M. F. Pollack of Union Linden High School, California, has retired.

Dr. D. H. Porter of Marion College has been promoted to the position of Professor of Mathematics and Physics.

Mr. W. O. Portmann has been appointed to a part-time instructorship at Case Institute of Technology.

Mr. L. B. Rall has been appointed to a research fellowship at Oregon State College.

Mr. H. L. Randolph, previously with the Bank of America, Long Beach, California, has a position as Engineering Analyst with Research-Garrett Corporation, Los Angeles, California.

Assistant Professor O. M. Rasmussen has been promoted to the position of Chairman of the Department of Mathematics, University of Denver.

Mr. W. A. Reid, recently a graduate student at Michigan State College, is now Applied Science Representative with International Business Machines, New York City.

Dean P. H. Renton of West Virginia Institute of Technology is teaching at Valley Regional High School, Deep River, Connecticut.

Assistant Professor C. L. Riggs of East Texas State Teachers College has been appointed to an associate professorship at Texas Technological College.

Miss Jean E. Sammet, formerly with Metropolitan Life Insurance Company, New York City, is now Assistant Project Engineer, Sperry Gyroscope Company, Great Neck, New York.

Mr. J. H. Sartain, previously with Edwin Shields Hewitt and Associates, Libertyville, Illinois, has accepted a position as Assistant Operations Analyst, Armour Research Foundation, Chicago, Illinois.

Dr. W. R. Seugling, formerly with Yoh Engineering Inc., Los Angeles, California, has a position as an engineering specialist with Allstates Engineering Company, Trenton, New Jersey.

Mr. F. A. C. Sevier, who has been a research mathematician at Bell Aircraft Corporation, Niagara Falls, New York, has been appointed Chairman of the Department of Mathematics of the College of South Jersey, Rutgers University.

Mrs. Dorothy B. Shaffer has been promoted to the position of Associate Mathematician at Cornell Aeronautical Laboratory, Buffalo, New York.

Mr. J. M. Shaheen has been appointed to an assistant professorship at Tri-State College.

Associate Professor L. W. Sheridan, recently a research engineer with the Minneapolis-Honeywell Regulator Company, Minnesota, has accepted a position as Senior Physicist with General Mills, Engineering Research and Development Division, Minneapolis, Minnesota.

Miss Bettie L. Shipman, formerly a teaching fellow at Texas Technological College, is teaching at Abernathy High School, Texas.

Mr. L. E. Sigler has been appointed Lecturer at Columbia University.

Mr. D. B. Singer has a position as Structural Research Engineer with Armour Research Foundation, Illinois Institute of Technology.

Mr. F. M. Sioson, previously at the Geophysical Observatory, Quezon City, Philippines, has been appointed to an instructorship at the College of Agriculture, University of the Philippines.

Sister Mary Stephanie of Georgian Court College has been promoted to an assistant professorship.

Dr. Michael Skalskyj of Xavier University has been promoted to an assistant professorship.

Mr. Abe Sklar, formerly a graduate student at the University of Chicago, has been appointed to a research assistantship at California Institute of Technology.

Associate Professor R. E. Smith of Duquesne University has been promoted to the position of Chairman of the Department of Mathematics.

Mr. T. C. Smith of Phillips University has been appointed to an instructorship in the Department of Engineering Mechanics, University of Nebraska.

Mr. R. L. Snider has been appointed to an instructorship at York Junior College, Pennsylvania.

Assistant Professor R. H. Sorgenfrey of the University of California at Los Angeles has been promoted to an associate professorship.

Mr. H. L. Steinberg has a position with Rich's, Stamp and Coin Department, Atlanta, Georgia.

Dr. R. G. Stoneham of the University of California has been appointed to an assistant professorship at San Diego State College.

Mr. P. Y. Tani, previously with Minneapolis-Honeywell Regulator Company, Minneapolis, has accepted a position as Research Engineer with North American Aviation, Los Angeles, California.

Mr. Liston Tatum, Jr., is now Sales Representative for International Business Machines, Cincinnati, Ohio.

Assistant Professor Takashi Terami of the College of St. Thomas has been promoted to an associate professorship.

Mr. G. T. Thompson, recently a graduate assistant at Oregon State College, has a position as Applied Mathematician at Jet Propulsion Laboratory, California Institute of Technology.

Associate Professor George Tunell of the Department of Geology, University of California at Los Angeles, has been promoted to a professorship.

Assistant Professor William Wallace of Northeastern University has been promoted to an associate professorship.

Mr. C. R. Wampole is now Head of the Department of Mathematics of Eastern Military Academy, Cold Spring Harbor, New York.

Dr. W. H. Warner, formerly a research assistant at Carnegie Institute of Technology, is now a research associate in the Graduate Division of Applied Mathematics, Brown University.

Dr. F. J. Weyl has been appointed Director of the Mathematical Sciences Division, United States Office of Naval Research, Washington, D. C.

Assistant Professor Herbert Wolf of the University of South Carolina has been appointed to an associate professorship at Tri-State College.

Mr. F. G. Wolfort, recently a student at St. John's University, has been appointed to a research assistantship at Massachusetts Institute of Technology.

The following appointments to graduate assistantships have been announced: Catholic University of America, Mr. D. B. J. Tomiuk; Lehigh University, Mr. E. K. Dorff; Ohio State University, Mr. P. J. Nikolai; Purdue University, Mr. R. E. Dowds, Mr. D. G. Johnson; University of Georgia, Mr. L. H. Williams; University of Skoplje, Yugoslavia, Miss Kovina Milosevich.

The following appointments to teaching assistantships have been announced: Cornell University, Mr. G. E. Collins; University of British Columbia, Mr. C. A. Swanson; University of California, Mr. Melvin Pomerantz; University of Oklahoma, Mr. J. E. Hoffman; University of Virginia, Mr. J. J. Greever, III.

Professor Emeritus E. M. Berry of State Teachers College, Chadron, Nebraska, died on April 26, 1953; he had been a member of the Association for thirty-three years.

Professor Emeritus G. R. Livingston of San Diego State College died on September 2, 1953; he had been a member of the Association for thirty-four years.

President Emeritus C. W. Smith of Wisconsin State College, Superior, died on November 10, 1953; he was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-SEVENTH ANNUAL MEETING OF THE ASSOCIATION

The thirty-seventh annual meeting of the Mathematical Association of America was held at Johns Hopkins University, Baltimore, Maryland on Thursday, December 31, 1953, in conjunction with the annual meeting of the American Mathematical Society. About six hundred and forty persons were registered, including the following three hundred and sixty members of the Association:

J. C. Abbott, M. I. Aissen, C. B. Allendoerfer, R. D. Anderson, H. A. Antosiewicz, K. J. Arnold, L. A. Aroian, Sholom Arzt, W. L. Ayres, Frances E. Baker, B. J. Ball, N. H. Ball, W. E. Barnes, I. A. Barnett, Grace E. Bates, E. G. Begle, W. J. Bellmer, T. J. Benac, A. A. Bennett, R. H. Bing, Z. W. Birnbaum, I. E. Block, I. L. Bossler, T. A. Botts, S. G. Bourne, Julia W. Bower, J. W. Brace, Leila D. Bram, H. W. Brinkmann, A. R. Brown, Jr., B. H. Buikstra, R. S. Burington, J. H. Bushey, Jewell H. Bushey, G. H. Butcher, S. S. Cairns, E. A. Cameron, R. H. Cameron, H. H. Campaigne, J. F. Canu, L. Virginia Carlton, Dorothy I. Carpenter, W. B. Carver, J. W. Cell, E. W. Cheney, P. L. Chessin, B. H. Chovitz, D. E. Christie, G. F. Clanton, C. E. Clark, F. E. Clark, F. Marion Clarke, G. R. Clements, A. H. Clifford, A. H. Cockshott, E. W. Coffin, E. J. Cogan, L. W. Cohen, Teresa Cohen, Nancy Cole, A. H. Copeland, J. B. Cornelison, W. H. H. Cowles, V. F. Cowling, J. H. Curtiss, Wayne Dancer, R. L. Davis, L. J. Deck, R. C. Di Prima, F. G. Dressel, Nelson Dunford, W. L. Duren, Jr., W. H. Durfee, W. F. Eberlein, R. P. Eddy, H. P. Edmundson, D. F. Eliezer, Joanne Elliott, Arthur Erdélyi, Paul Erdös, D. H. Erkiletian, Jr., W. H. Fagerstrom, William Feller, F. A. Ficken, N. J. Fine, W. T. Fishback, Harley Flanders, B. A. Fleishman, S. A. Foote, Gloria C. Ford, L. R. Ford, M. K. Fort, Jr., Tomlinson Fort, A. H. Fox, J. S. Frame, C. H. Frick, R. E. Fullerton, H. M. Gehman, J. J. Gehrig, J. J. Gergen, B. C. Getchell, K. G. Getman, Wallace Givens, A. M. Gleason, Marion Goddard, H. W. Godderz, Casper Goffman, V. D. Gokhale, Michael Goldberg, Oscar Goldman, W. A. Golomski, R. A. Good, A. W. Goodman, W. H. Gottschalk, E. C. Gras, F. L. Griffin, J. S. Griffin, Jr., Emil Grosswald, Arnold Grudin, Samuel Gruenzweig, P. E. Guenther, V. H. Haag, Franklin Haimo, E. E. Hammond, Jr., J. R. Hammond, H. W. Handsfield, H. H. Hartzler, Melvin Hausner, E. A. Hedberg, Marguerite Z. Hedberg, G. A. Hedlund, C. E. Heilman, G. C. Helme, Melvin Henriksen, R. T. Herbst, Coleman Herpel, I. R. Hershner, Jr., I. N. Herstein, P. S. Herwitz, I. I. Hirschman, A. J. Hoffman, J. E. Hoffman, F. E. Hohn, Betty W. Holz, J. R. Holzinger, J. G. Horne, Jr., E. Marie Hove, G. B. Huff, Ralph Hull, D. W. Hullinghorst, Louise S. Hunter, W. A. Hurwitz, M. A. Hyman, R. F. Jackson, S. B. Jackson, H. G. Jacob, Jr., C. A. Johnson, R. E. Johnson, B. W. Jones, F. B. Jones, John Jones, Jr., Mark Kac, Robert Kalin, Hyman Kamel, L. M. Kells, J. B. Kelly, L. M. Kelly, J. H. B. Kemperman, M. R. Kenner, J. R. F. Kent, D. E. Kibbey, E. R. Kiely, H. L. Kinsolving, George Klein, J. R. Kline, Rev. C. F. Koehler, Fulton Koehler, T. L. Koehler, H. L. Krall, R. R. Kuebler, Jr., Stephen Kulik, Harry Langman, Leo Lapidus, Rev. E. H. Larguier, C. G. Latimer, Solomon Lefschetz, Marguerite Lehr, R. A. Leibler, Walter Leighton, R. B. Leipnik, Anne L. Lewis, D. C. Lewis, Jr., E. V. Lewis, F. W. Light, Jr., D. B. Lloyd, Charles Loewner, Lee Lorch, F. W. Lott, Jr., D. B. Lowdenslager, L. L. Lowenstein, C. I. Lubin, Elizabeth C. Lukacs, R. C. Lyndon, Rev. J. J. MacDonnell, C. C. MacDuffee, G. R. MacLane, Saunders MacLane, H. M. MacNeille, W. G. Madow, J. F. Manogue, M. H. Martin, A. P. Mattuck, K. O. May, V. O. McBrien, B. H. McCandless, Sophia L. McDonald, Edith A. McDougle, S. S. McNeary, E. J. McShane, Florence M. Mears, A. E. Meder, Jr., Herman Meyer, Joseph Milkman, D. D. Miller, F. H. Miller, A. K. Mitchell, Deane Montgomery, A. H. Moore, T. W. Moore, W. K. Morrill, C. W. Munshower, F. D. Murnaghan, C. H. Murphy, Jr., W. R. Murray, Morris Newman, C. V. Newsom, R. A. Niemann, A. B. J. Novikoff, C. O. Oakley, Ruth E. O'Donnell, M. W. Oliphant,

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Sessions of the Association were held on Thursday morning and afternoon in Remsen Hall of Johns Hopkins University, with President E. J. McShane presiding. The Program Committee for the meeting consisted of W. G. Madow, Chairman; David Blackwell, and E. P. Northrop.

INTRODUCTORY NOTE ON THE PROGRAM

The papers which were presented had largely resulted from various projects intended to improve undergraduate mathematical education or to provide a more satisfactory transition from school to college.

Professor Saunders MacLane who was president of the Association in 1951 and 1952, has had a deep interest in attempts to improve mathematical education, and was active in organizing some of the projects which are mentioned below.

Two of the papers dealt with the transition from school to college. Professor H. W. Brinkmann is chairman of the Subcommittee on Mathematics of the School and College Study of Admission with Advanced Standing sponsored by twelve colleges and universities. Professor E. G. Begle was a member of the Mathematics Panel of the School and College Study of General Education sponsored by three schools and three universities. Both of these studies were aided by grants from the Fund for the Advancement of Education established by the Ford Foundation.

As part of its continuing program in the mathematical training of social scientists, the Social Science Research Council sponsored an eight-week intensive summer session at Dartmouth College, known as the 1953 Summer Institute in Mathematics for Social Scientists. All students in this Institute were college graduates; about half had received their doctorates in a social science. A large

part of the curriculum consisted of topics, usually taught in advanced undergraduate or beginning graduate courses in mathematics, that were believed to be suitable for undergraduate mathematical training. The program was conducted with the help of a grant from the Behavioral Sciences Division of the Ford Foundation.

The session on New College Mathematics was the result of three related projects. Professors W. L. Duren, Jr., G. B. Price, and A. W. Tucker are members of the Association's Committee on the Undergraduate Program in Mathematics. Dr. C. V. Newsom is chairman of the Joint Committee on Teacher Training in Mathematics sponsored by the Mathematical Association of America and the National Council of Teachers of Mathematics. Professor B. W. Jones was director of an eight-week Summer Conference in Collegiate Mathematics that met at the University of Colorado under the sponsorship of the National Science Foundation and the University of Colorado.

FIRST SESSION OF THE ASSOCIATION

Round Table: "Mathematics for Social Scientists," by the faculty of the 1953 Summer Institute: Professor W. G. Madow, University of Illinois, Professor R. M. Thrall, University of Michigan, Professor R. R. Bush, Harvard University, and Professor Howard Raiffa, Columbia University.

Retiring Presidential Address: "Of Course and Courses," by Professor Saunders MacLane, University of Chicago.

SECOND SESSION OF THE ASSOCIATION

"Secondary School Mathematics for the Exceptional Student," by Professor H. W. Brinkmann, Swarthmore College.

"A Study of the Integration of School and College Mathematics," by Professor E. G. Begle, Yale University.

New College Mathematics

A symposium exploring some aspects of the problems presented by efforts to introduce major reforms into the undergraduate program in mathematics.

Introductory remarks, by Professor G. B. Price, University of Kansas.

"What should we teach in college mathematics?," by Dr. C. V. Newsom, Associate Commissioner for Higher Education, State of New York.

"What is the role of the set concepts?," by Professor W. L. Duren, Jr., Tulane University.

"What kind of mathematics will social scientists need?," by Professor A. W. Tucker, Princeton University.

"Can our present teachers convert to a new program? A report on the 1953 Summer Conference at Boulder," by Professor B. W. Jones, University of Colorado.

MEETING OF THE BOARD OF GOVERNORS

The Board met on Wednesday morning in the Lounge of Levering Hall, with seventeen members present. Among the more important items of business transacted were the following:

Professor J. F. Randolph of the University of Rochester was re-elected as a member of the Finance Committee for the four-year term 1954–1957.

Approval was given to the appointment by President McShane of the following Nominating Committee for 1954: A. L. Putnam, Chairman; P. S. Jones, and E. E. Moise.

It was voted to hold the thirty-eighth annual meeting at the University of Pittsburgh, Pittsburgh, Pennsylvania, on Thursday, December 30, 1954.

On the recommendation of the Finance Committee, the Board voted that $500 be transferred from the General Fund to the Chauvenet Fund. It is expected that this will leave a sufficiently large balance in the Chauvenet Fund, so that hereafter the payments of the Chauvenet Prize may be made from income alone.

The Board also voted that the expenses of the second edition of "Professional Opportunities in Mathematics" be charged against the Chace Fund and that sales of this pamphlet be credited to the Chace Fund.

Professor J. S. Frame reported for the Committee on Employment Opportunities that the Council of the American Mathematical Society had also voted its approval of the establishment of a register of positions open to mathematicians at the joint meetings of the Society and the Association. The first such register was available at Levering Hall on Wednesday afternoon and Thursday.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Thursday, December 31, 1953, at 2:00 P.M. in Remsen Hall of Johns Hopkins University, Baltimore, Maryland. Vice-President F. L. Griffin presided.

The Secretary announced the results of the balloting for officers, in which 1486 votes were cast. H. S. M. Coxeter of the University of Toronto was elected First Vice-President for the two-year term 1954–1955. Philip Franklin of the Massachusetts Institute of Technology and C. E. Springer of the University of Oklahoma were elected Governors for the three-year term 1954–1956.

As chairman of the Committee on the 1953 Chauvenet Prize, Professor C. B. Allendoerfer presented the 1953 Chauvenet Prize to Professor E. J. McShane of the University of Virginia for his paper, "Partial Orderings and Moore-Smith Limits," published in this MONTHLY, volume 59, 1952, pp. 1–11.

MEETINGS OF OTHER ORGANIZATIONS

The sessions of the American Mathematical Society began on Monday, December 28 and continued through Wednesday afternoon. The Josiah Willard Gibbs Lecture entitled "Mathematics in Economics" was delivered on Monday

evening by Professor Wassily Leontief of Harvard University. Invited addresses were delivered by Professors A. D. Wallace of Tulane University and D. V. Widder of Harvard University.

The Pi Mu Epsilon Fraternity held its annual meeting on Monday in Levering Hall.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of D. C. Lewis, Chairman; W. L. Chow, A. H. Clifford, L. W. Cohen, H. M. Gehman, E. K. Haviland, W. K. Morrill, Marian M. Torrey.

Registration took place in Levering Hall on the Homewood Campus of Johns Hopkins University. Rooms were available in the Alumni Memorial Dormitory and at the Bradford Apartments from Monday morning until Friday morning. Hotel accommodations were also available in downtown Baltimore. Meals were served in the cafeteria of Levering Hall and at the Faculty Club.

The ladies of the Department of Mathematics of Johns Hopkins University entertained at tea in the main ballroom of Levering Hall on Monday afternoon. Sightseeing tours of Annapolis and of Baltimore were held on Tuesday and Wednesday, respectively.

The banquet for the members of the mathematical organizations and their guests was held on Wednesday evening in Levering Hall. Professor D. C. Lewis, acting as toastmaster, welcomed those in attendance on behalf of Johns Hopkins University. President E. J. McShane of the Association and President G. T. Whyburn of the Society responded on behalf of their respective organizations. Dr. Alan T. Waterman, Director of the National Science Foundation, described the program of the Foundation for grants and scholarships in the field of Mathematics.

Professor Ralph Hull presented a resolution of thanks which was adopted by a hearty and unanimous vote. The resolution expressed appreciation to the members of the local committee and to the authorities of Johns Hopkins University for their efforts in arranging this enjoyable meeting.

H. M. GEHMAN, *Secretary-Treasurer*

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On the U. S. Subcommittee of the International Mathematical Instruction Committee (I.M.U.K.):
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EMPLOYMENT OPPORTUNITIES

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The MONTHLY is devoting this space to paid announcements of employment opportunities for mathematicians. The text of such announcements should be in want-ad form and must be in the hands of the editor (C. B. Allendoerfer, Mathematics Department, University of Washington, Seattle 5, Wash.) before the first day of the month preceding the issue in which the notice is to appear. Announcements should indicate the academic rank or similar description of the opening, but should not mention a specific salary. Blind ads are permissible which direct replies to a specific box number in care of the Mathematical Association of America, Buffalo 14, N. Y. In order to conserve space and achieve uniformity, the privilege is reserved to rearrange advertisements. Advertisers will be billed by the Association at the rate of $1.50 per line. Rates for display advertising may be obtained from the Advertising Manager.

CALENDAR OF FUTURE MEETINGS

Thirty-fifth Summer Meeting, University of Wyoming, Laramie, Wyoming, August 30–31, 1954.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Marshall College, Huntington, West Virginia, May 1, 1954.
ILLINOIS, Knox College, Galesburg, May 14–15, 1954.
INDIANA, Rose Polytechnic Institute, Terre Haute, May 1, 1954.
IOWA, Iowa State College, Ames, April 30–May 1, 1954.
KANSAS, Baker University, Baldwin City, March 27, 1954.
KENTUCKY, University of Kentucky, April 24, 1954.
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, St. John's University, Brooklyn, March 27, 1954.
MICHIGAN, University of Michigan, Ann Arbor, March 27, 1954.
MINNESOTA, Hamline University, St. Paul, May 8, 1954.
MISSOURI, University of Missouri, Columbia, May 7, 1954.
NEBRASKA, Omaha, April 24, 1954.
NORTHERN CALIFORNIA
OHIO, Ohio State University, Columbus, April 17, 1954.
OKLAHOMA, Oklahoma City University, October, 1954.
PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 18, 1954.
PHILADELPHIA
ROCKY MOUNTAIN, Colorado Agricultural and Mechanical College, Fort Collins, April 30–May 1, 1954.
SOUTHEASTERN, University of South Carolina, Columbia, March 19–20, 1954.
SOUTHERN CALIFORNIA, George Pepperdine College, Los Angeles, March 13, 1954.
SOUTHWESTERN, Arizona State College, Tempe, April 16–17, 1954.
TEXAS, Texas Technological College, Lubbock, April 23–24, 1954.
UPPER NEW YORK STATE, College for Teachers at Albany, May 1, 1954.
WISCONSIN, State Teachers College, Eau Claire, May 8, 1954.

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